Realizability for Constructive Zermelo-Fraenkel Set Theory

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Abstract

Constructive Zermelo-Fraenkel Set Theory, **CZF**, has emerged as a standard reference theory that relates to constructive predicative mathematics as **ZFC** relates to classical Cantorian mathematics. A hallmark of this theory is that it possesses a type-theoretic model. Aczel showed that it has a formulae-as-types interpretation in Martin-Löf's intuitionist theory of types [14, 15]. This paper, though, is concerned with a rather different interpretation. It is shown that Kleene realizability provides a self-validating semantics for **CZF**, viz. this notion of realizability can be formalized in **CZF** and demonstrably in **CZF** it can be verified that every theorem of **CZF** is realized.

This semantics, then, is put to use in establishing several equiconsistency results. Specifically, augmenting **CZF** by well-known principles germane to Russian constructivism and Brouwer's intuitionism turns out to engender theories of equal proof-theoretic strength with the same stock of provably recursive functions.

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1 Introduction

Realizability semantics for intuitionistic theories were first proposed by Kleene in 1945 [12]. Inspired by Kreisel's and Troelstra's [13] definition of realizability for higher order Heyting arithmetic, realizability was first applied to systems of set theory by Myhill [17] and Friedman [11]. More recently, realizability models of set theory were investigated by Beeson [6, 7] (for non-extensional set theories) and McCarty [16] (directly for extensional set theories). [16] is concerned with realizability for intuitionistic Zermelo-Fraenkel set theory, **IZF**, and employs transfinite iterations of the powerset operation through all the ordinals in defining the realizability (class) structure $\mathrm{V}(\mathcal{A})$ over any applicative structure $\mathcal{A}$. Moreover, in addition to the powerset axiom the approach in [16] also avails itself of

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unfettered separation axioms. At first blush, this seems to render the approach unworkable for **CZF** as this theory lacks the powerset axiom and has only bounded separation. However, it will be shown that these obstacles can be overcome.

Once one has demonstrated how to define $\mathrm{V}(\mathcal{A})$ on the basis of **CZF** there still remains the task of verifying that $\mathrm{V}(\mathcal{A})$ validates all the theorems of **CZF** when assuming just the axioms of **CZF** in the ground model. In particular the subset collection axiom poses a new challenge. Another interesting axiom that has been considered in the context of **CZF** is the regular extension axiom, **REA**. It will shown that **REA** holds in $\mathrm{V}(\mathcal{A})$ if it holds in the background universe. The pattern propagates when it comes to forms of the axiom of choice. Taking the standard applicative structure Kl based on Turing machine application, either of the axioms of countable choice, dependent choices, and the presentation axiom **PAx** propagate to $\mathrm{V}(Kl)$ if they hold in the underlying universe. This also improves on the proof of $\mathrm{V}(Kl) \models$ **PAx** in [16] which assumes the unrestricted axiom of choice in the ground model.

The most interesting applications of $\mathrm{V}(Kl)$ concern principles germane to Russian constructivism and Brouwer's intuitionism that are classically refutable. For example, Church's thesis, the uniformity principle, Unzerlegbarkeit, and the assertion that every function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is continuous hold in $\mathrm{V}(Kl)$. As a corollary, therefore, we obtain that augmenting **CZF** by these "exotic" axioms neither increases the proof-theoretic strength nor the stock of provably recursive functions. Drawing on interpretations of **CZF** and **CZF** + **REA** in classical Kripke-Platek set theories **KP** and **KPi**, respectively, it is also shown that Markov's principle and the principle of independence of premisses may be added without changing the outcome.

The plan for the paper is as follows: Section 1.1 reviews the axioms of **CZF** while section 1.2 recalls some axioms of choice. Section 2 provides the background on applicative structures which is put to use in section 3 to define the general realizability structure, $\mathrm{V}(\mathcal{A})$. Section 4 introduces the notion of realizability. Section 5 is devoted to showing that the axioms of **CZF** hold in the realizability structure. The validity of the regular extension in $\mathrm{V}(\mathcal{A})$ is proved in section 6. Markov's principle and the principle of independence of premisses axioms are discussed in section 7 and are shown to not increase the proof-theoretic strength. In section 8 we briefly discuss absoluteness properties between the background universe and $\mathrm{V}(Kl)$. Section 9 is devoted to principles germane to Russian constructivism and Brouwer's intuitionism that hold in $\mathrm{V}(Kl)$ while section 10 is concerned with choice principles in $\mathrm{V}(Kl)$. The last section 11 addresses Brouwerian continuity principles that hold in $\mathrm{V}(Kl)$.

1.1 The system CZF

In this subsection we will summarize the language and axioms for **CZF**. The language of **CZF** is the same first order language as that of classical Zermelo-Fraenkel Set Theory, **ZF** whose only non-logical symbol is $\in$. The logic of **CZF** is intuitionistic first order logic with equality. Among its non-logical axioms are *Extensionality*, *Pairing* and *Union* in their usual forms. **CZF** has additionally axiom schemata which we will now proceed to summarize.

Infinity: $\exists x \forall u[u \in x \leftrightarrow (\emptyset = u \vee \exists v \in x\, u = v+1)]$ where $v+1 = v \cup \{v\}$.

Set Induction: $\forall x[\forall y \in x \phi(y) \rightarrow \phi(x)] \rightarrow \forall x \phi(x)$

Bounded Separation: $\forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge \phi(x)]$

for all *bounded* formulae ϕ. A set-theoretic formula is *bounded* or *restricted* if it is constructed from prime formulae using $\neg, \wedge, \vee, \rightarrow, \forall x \in y$ and $\exists x \in y$ only.

Strong Collection: For all formulae ϕ,

$$\forall a[\forall x \in a \exists y \phi(x,y) \;\rightarrow\; \exists b\, [\forall x \in a\, \exists y \in b\, \phi(x,y) \wedge \forall y \in b\, \exists x \in a\, \phi(x,y)]].$$

Subset Collection: For all formulae ψ,

$$\begin{aligned}&\forall a \forall b \exists c \forall u\, [\forall x \in a\, \exists y \in b\; \psi(x,y,u) \;\rightarrow \\ &\quad \exists d \in c\, [\forall x \in a\, \exists y \in d\, \psi(x,y,u) \wedge \forall y \in d\, \exists x \in a\, \psi(x,y,u)]].\end{aligned}$$

The Subset Collection schema easily qualifies as the most intricate axiom of **CZF**. To explain this axiom in different terms, we introduce the notion of *fullness* (cf. [1]).

Definition: 1.1 As per usual, we use $\langle x, y \rangle$ to denote the ordered pair of x and y. We use $\mathbf{Fun}(g)$, $\mathbf{dom}(R)$, $\mathbf{ran}(R)$ to convey that g is a function and to denote the domain and range of any relation R, respectively.

For sets A, B let $A \times B$ be the cartesian product of A and B, that is the set of ordered pairs $\langle x, y \rangle$ with $x \in A$ and $y \in B$. Let ${}^{A}B$ be the class of all functions with domain A and with range contained in B. Let $\mathbf{mv}({}^{A}B)$ be the class of all sets $R \subseteq A \times B$ satisfying $\forall u \in A\, \exists v \in B\, \langle u, v \rangle \in R$. A set C is said to be *full in* $\mathbf{mv}({}^{A}B)$ if $C \subseteq \mathbf{mv}({}^{A}B)$ and

$$\forall R \in \mathbf{mv}({}^{A}B)\, \exists S \in C\, S \subseteq R.$$

The expression $\mathbf{mv}({}^{A}B)$ should be read as the collection of *multi-valued functions* from the set A to the set B.

Additional axioms we shall consider are:

Exponentiation: $\forall x \forall y \exists z\, z = {}^{x}y$.

Fullness: $\forall x \forall y \exists z$ *z is full in* $\mathbf{mv}({}^{x}y)$.

The next result provides an equivalent rendering of Subset Collection.

Proposition: 1.2 *Let* $\mathbf{CZF}^{-}$ *be* **CZF** *without* Subset Collection.

(i) $\mathbf{CZF}^{-} \vdash$ *Subset Collection* $\leftrightarrow$ *Fullness.*

(ii) **CZF** $\vdash$ *Exponentiation.*

Proof: [1], 2.2. ☐

Let **EM** be the principle of excluded third, i.e. the schema consisting of all formulae of the form $\theta \vee \neg\theta$. The first central fact to be noted about **CZF** is:

Proposition: 1.3 **CZF** + **EM** = **ZF**.

Proof: Note that classically Collection implies Separation. Powerset follows classically from Exponentiation. ☐

On the other hand, it was shown in [20], Theorem 4.14, that **CZF** has only the strength of Kripke-Platek Set Theory (with the Infinity Axiom), **KP** (see [5]), and, moreover, that **CZF** is of the same strength as its subtheory $\mathbf{CZF}^{-}$, i.e., **CZF** minus Subset Collection. To stay in the world of **CZF** one has to keep away from any principles that imply **EM**. Moreover, it is perhaps fair to say that **CZF** is such an interesting theory owing to the non-derivability of Powerset and Separation. Therefore one ought to avoid any principles which imply Powerset or Separation.

The first large set axiom proposed in the context of constructive set theory was the *Regular Extension Axiom*, **REA**, which Aczel introduced to accommodate inductive definitions in **CZF** (cf. [3]).

Definition: 1.4 A is inhabited if $\exists x\, x \in A$. An inhabited set A is *regular* if A is transitive, and for every $a \in A$ and set $R \subseteq a \times A$ if $\forall x \in a\, \exists y\, (\langle x, y\rangle \in R)$, then there is a set $b \in A$ such that

$$\forall x \in a\, \exists y \in b\, (\langle x, y\rangle \in R) \;\wedge\; \forall y \in b\, \exists x \in a\, (\langle x, y\rangle \in R).$$

In particular, if $R : a \to A$ is a function, then the image of R is an element of A.

The *Regular Extension Axiom*, **REA**, is as follows: *Every set is a subset of a regular set.*

1.2 Axioms of choice

In many a text on constructive mathematics, axioms of countable choice and dependent choices are accepted as constructive principles. This is, for instance, the case in Bishop's constructive mathematics (cf.[8]) as well as Brouwer's intuitionistic analysis (cf.[22], Ch.4, Sect.2). Myhill also incorporated these axioms in his constructive set theory [18].

The weakest constructive choice principle we shall consider is $\mathbf{AC}^{\omega,\omega}$ which asserts that whenever $\forall i{\in}\omega\, \exists j{\in}\omega\, \theta(i, j)$ then there exists a function $f : \omega \to \omega$ such that $\forall i{\in}\omega\, \theta(i, f(i))$.

The *Axiom of Countable Choice*, $\mathbf{AC}_{\omega}$, is the following scheme: whenever $\forall i{\in}\omega\, \exists x\, \psi(i, x)$ then there exists a function f with domain ω such that $\forall i{\in}\omega\, \theta(i, f(i))$. Obviously $\mathbf{AC}_{\omega}$ implies $\mathbf{AC}^{\omega,\omega}$.

A mathematically very useful axiom to have in set theory is the *Dependent Choices Axiom*, **DC**, i.e., for all formulae ψ, whenever

$$(\forall x{\in}a)\, (\exists y{\in}a)\, \psi(x, y)$$

and $b_0 \in a$, then there exists a function $f : \omega \to a$ such that $f(0) = b_0$ and

$$(\forall n \in \omega)\, \psi(f(n), f(n+1)).$$

Even more useful is the *Relativized Dependent Choices Axiom*, **RDC**. It asserts that for arbitrary formulae ϕ and ψ, whenever

$$\forall x[\phi(x) \;\to\; \exists y(\phi(y) \;\wedge\; \psi(x,y))]$$

and $\phi(b_0)$, then there exists a function f with domain ω such that $f(0) = b_0$ and

$$(\forall n \in \omega)[\phi(f(n)) \;\wedge\; \psi(f(n), f(n+1))].$$

We shall use the notation $f : X \twoheadrightarrow Y$ to convey that f is a function from X onto Y. A set P is a *base* if for any P-indexed family $(X_a)_{a \in P}$ of inhabited sets X_a, there exists a function f with domain P such that, for all $a \in P$, $f(a) \in X_a$. The *Presentation Axiom*, **PAx**, is the statement that every set is the surjective image of a base, i.e., for all sets A there exists a base B and a function $f : B \twoheadrightarrow A$.

2 Some background on applicative structures

In order to define a realizability interpretation we must have a notion of realizing functions on hand. A particularly general and elegant approach to realizability builds on structures which have been variably called *partial combinatory algebras*, *applicative structures*, or *Schönfinkel algebras*. These structures are best described as the models of a theory **APP**. The following presents the main features of **APP**; for full details cf. [9, 10, 7, 22]. The language of **APP** is a first-order language with a ternary relation symbol App, a unary relation symbol N (for a copy of the natural numbers) and equality, $=$, as primitives. The language has an infinite collection of variables, denoted $x, y, z, \ldots$, and nine distinguished constants: $\mathbf{0}, \mathbf{s}_N, \mathbf{p}_N, \mathbf{k}, \mathbf{s}, \mathbf{d}, \mathbf{p}, \mathbf{p_0}, \mathbf{p_1}$ for, respectively, zero, successor on N, predecessor on N, the two basic combinators, definition by cases, pairing and the corresponding two projections. There is no arity associated with the various constants. The *terms* of **APP** are just the variables and constants. We write $t_1 t_2 \simeq t_3$ for $\mathrm{App}(t_1, t_2, t_3)$.

Formulae are then generated from atomic formulae using the propositional connectives and the quantifiers.

In order to facilitate the formulation of the axioms, the language of **APP** is expanded definitionally with the symbol $\simeq$ and the auxiliary notion of an *application term* is introduced. The set of application terms is given by two clauses:

1. all terms of **APP** are application terms; and
2. if s and t are application terms, then (st) is an application term.

For s and t application terms, we have auxiliary, defined formulae of the form:

$$s \simeq t \quad := \quad \forall y(s \simeq y \leftrightarrow t \simeq y),$$

if t is not a variable. Here $s \simeq a$ (for a a free variable) is inductively defined by:

$$s \simeq a \quad \text{is} \quad \begin{cases} s = a, & \text{if } s \text{ is a term of } \mathbf{APP}, \\ \exists x, y[s_1 \simeq x \wedge s_2 \simeq y \wedge \mathrm{App}(x, y, a)] & \text{if } s \text{ is of the form } (s_1 s_2). \end{cases}$$

Some abbreviations are $t_1 t_2 \dots t_n$ for $((...(t_1 t_2)...)t_n)$; $t \downarrow$ for $\exists y(t \simeq y)$ and $\phi(t)$ for $\exists y(t \simeq y \wedge \phi(y))$.

Some further conventions are useful. Systematic notation for *n-tuples* is introduced as follows: (t) is t, (s, t) is $\mathbf{p}st$, and $(t_1, \dots, t_n)$ is defined by $((t_1, \dots, t_{n-1}), t_n)$. In this paper, the **logic** of **APP** is assumed to be that of intuitionistic predicate logic with identity. **APP**'s **non-logical axioms** are the following:

Applicative Axioms

1. $\mathrm{App}(a, b, c_1) \wedge \mathrm{App}(a, b, c_2) \rightarrow c_1 = c_2$.
2. $(\mathbf{k}ab) \downarrow \wedge \mathbf{k}ab \simeq a$.
3. $(\mathbf{s}ab) \downarrow \wedge \mathbf{s}abc \simeq ac(bc)$.
4. $(\mathbf{p}a_0 a_1) \downarrow \wedge (\mathbf{p_0}a) \downarrow \wedge (\mathbf{p_1}a) \downarrow \wedge \mathbf{p_i}(\mathbf{p}a_0 a_1) \simeq a_i$ for $i = 0, 1$.
5. $N(c_1) \wedge N(c_2) \wedge c_1 = c_2 \rightarrow \mathbf{d}abc_1c_2 \downarrow \wedge \mathbf{d}abc_1c_2 \simeq a$.
6. $N(c_1) \wedge N(c_2) \wedge c_1 \neq c_2 \rightarrow \mathbf{d}abc_1c_2 \downarrow \wedge \mathbf{d}abc_1c_2 \simeq b$.
7. $\forall x\, (N(x) \rightarrow [\mathbf{s}_N x \downarrow \wedge \mathbf{s}_N x \neq \mathbf{0} \wedge N(\mathbf{s}_N x)])$.
8. $N(\mathbf{0}) \wedge \forall x\, (N(x) \wedge x \neq \mathbf{0} \rightarrow [\mathbf{p}_N x \downarrow \wedge \mathbf{s}_N(\mathbf{p}_N x) = x])$.
9. $\forall x\, [N(x) \rightarrow \mathbf{p}_N(\mathbf{s}_N x) = x]$
10. $\varphi(\mathbf{0}) \wedge \forall x[N(x) \wedge \varphi(x) \rightarrow \varphi(\mathbf{s}_N x)] \rightarrow \forall x[N(x) \rightarrow \varphi(x)]$.

Let $\mathbf{1} := \mathbf{s}_N\, \mathbf{0}$. The foregoing applicative axioms entail that $\mathbf{1}$ is an application term that evaluates to an object, i.e., $\mathbf{1} \downarrow$, and that $N(\mathbf{1})$ as well as $\mathbf{0} \neq \mathbf{1}$ hold.

Employing the axioms for the combinators **k** and **s** one can deduce an abstraction lemma yielding λ-terms of one argument. This can be generalized using n–tuples and projections.

Lemma: 2.1 (cf. [9]) *(***Abstraction Lemma***) For each application term t there is a new application term t^* such that the parameters of t^* are among the parameters of t minus $x_1, \dots, x_n$ and such that*

$$\mathbf{APP} \vdash t^* \downarrow \wedge\ t^*(x_1, \dots, x_n) \simeq t.$$

$\lambda(x_1, \dots, x_n).t$ is written for t^*.

The most important consequence of the Abstraction Lemma is the Recursion Theorem. It can be derived in the same way as for the λ–calculus (cf. [9], [10], [7], VI.2.7). Actually, one can prove a uniform version of the following in **APP**.

Corollary: 2.2 (Recursion Theorem)

$$\forall f \exists g \forall x_1 \dots \forall x_n \, g(x_1, \dots, x_n) \simeq f(g, x_1, \dots, x_n).$$

The "standard" applicative structure is Kl in which the universe $|Kl|$ is ω and $\mathrm{App}^{Kl}(x, y, z)$ is Turing machine application:

$$\mathrm{App}^{Kl}(x, y, z) \quad \text{iff} \quad \{x\}(y) \simeq z.$$

The primitive constants of **APP** are interpreted over $|Kl|$ in the obvious way.

3 The general realizability structure

The following discussion assumes that we can formalize the notion of an applicative structure in **CZF**. Moreover, for the remainder of this paper, $\mathcal{A}$ will be assumed to be a fixed but arbitrary applicative structure, which in particular is a set.

The definition of the following realizability structure is due to McCarty [16].

Definition: 3.1 Ordinals are transitive sets whose elements are transitive also. We use lower case Greek letters to range over ordinals. For $\mathcal{A} \models$ **APP**,

$$\mathrm{V}(\mathcal{A})_\alpha = \bigcup_{\beta \in \alpha} \mathcal{P}(|\mathcal{A}| \times \mathrm{V}(\mathcal{A})_\beta). \tag{1}$$

$$\mathrm{V}(\mathcal{A}) = \bigcup_{\alpha} \mathrm{V}(\mathcal{A})_\alpha. \tag{2}$$

As the power set operation is not available in **CZF** it is not clear whether the universe $\mathrm{V}(\mathcal{A})$ can be formalized in **CZF**. To show this we shall review some facts showing that **CZF** accommodates inductively defined classes.

3.1 Inductively defined classes in CZF

Definition: 3.2 An *inductive definition* is a class of ordered pairs. If Φ is an inductive definition and $\langle x, a \rangle \in \Phi$ then we write

$$\frac{x}{a}\,\Phi$$

and call $\frac{x}{a}\,\Phi$ an *(inference) step* of Φ, with set x of *premisses* and *conclusion* a. For any class Y, let

$$\Gamma_\Phi(Y) = \{a : \exists x \, (x \subseteq Y \;\wedge\; \frac{x}{a}\,\Phi\,)\}.$$

The class Y is Φ*-closed* if $\Gamma_\Phi(Y) \subseteq Y$. Note that Γ_Φ is monotone; i.e. for classes Y_1, Y_2, whenever $Y_1 \subseteq Y_2$, then $\Gamma_\Phi(Y_1) \subseteq \Gamma_\Phi(Y_2)$.

We define the class *inductively defined by* Φ to be the smallest Φ-closed class. The main result about inductively defined classes states that this class, denoted $\mathbf{I}(\Phi)$, always exists.

Lemma: 3.3 (**CZF**) (Class Inductive Definition Theorem) *For any inductive definition* Φ *there is a smallest* Φ*-closed class* $\mathbf{I}(\Phi)$.

Moreover, there is a class $J \subseteq \mathbf{ON} \times V$ *such that*

$$\mathbf{I}(\Phi) \;=\; \bigcup_\alpha J^\alpha,$$

and for each α*,*

$$J^\alpha \;=\; \Gamma_\Phi(\bigcup_{\beta\in\alpha} J^\beta).$$

J *is uniquely determined by the above, and its stages* J^α *will be denoted by* Γ^α_Φ*.*

Proof: [2], section 4.2 or [4], Theorem 5.1. □

Lemma: 3.4 *The classes* $\mathrm{V}(\mathcal{A})_\alpha$ *are definable in* **CZF***.*

Proof: Let Φ be the inductive definition with

$$\frac{x}{a}\,\Phi \quad \text{iff} \quad \forall u{\in}a\; (u \in |\mathcal{A}| \times x).$$

Invoking Lemma 3.3, let J be the class such that $\mathbf{I}(\Phi) = \bigcup_\alpha J^\alpha$, and for each α,

$$J^\alpha \;=\; \Gamma_\Phi(\bigcup_{\beta\in\alpha} J^\beta).$$

Now let

$$\Delta_\alpha \;:=\; \bigcup_{\beta\in\alpha} J^\beta.$$

Note that $\Gamma_\Phi(X) = \mathcal{P}(|\mathcal{A}| \times X)$, and therefore

$$\begin{aligned} \Delta_\alpha &= \bigcup_{\beta\in\alpha} J^\beta \qquad\qquad (3)\\ &= \bigcup_{\beta\in\alpha} \Gamma_\Phi(\bigcup_{\eta\in\beta} J^\eta)\\ &= \bigcup_{\beta\in\alpha} \mathcal{P}(|\mathcal{A}| \times \bigcup_{\eta\in\beta} J^\eta)\\ &= \bigcup_{\beta\in\alpha} \mathcal{P}(|\mathcal{A}| \times \Delta_\beta). \end{aligned}$$

Letting $\mathrm{V}(\mathcal{A})_\alpha := \Delta_\alpha$, (3) shows that the equations of definition 3.1 obtain. □

Lemma: 3.5 (**CZF**).

(i) $\mathrm{V}(\mathcal{A})$ *is cumulative: for* $\beta \in \alpha$, $\mathrm{V}(\mathcal{A})_\beta \subseteq \mathrm{V}(\mathcal{A})_\alpha$.

(ii) If b *is a set such that* $b \subseteq |\mathcal{A}| \times \mathrm{V}(\mathcal{A})$ *then* $b \in \mathrm{V}(\mathcal{A})$.

Proof: (i) is immediate by (1).

For (ii), suppose $b \subseteq |\mathcal{A}| \times \mathrm{V}(\mathcal{A})$. Then

$$\forall x \in b\, \exists \alpha\, \exists e \in |\mathcal{A}|\, \exists z \in \mathrm{V}(\mathcal{A})_\alpha\ x = \langle e, z \rangle,$$

and therefore by Strong Collection there exists a set D such that

$$\forall x \in b\, \exists \alpha \in D\, \exists e \in |\mathcal{A}|\, \exists z \in \mathrm{V}(\mathcal{A})_\alpha\ x = \langle e, z \rangle,$$

where D is a set of ordinals. Now let $D' = \{\alpha + 1 : \ \alpha \in D\}$ and $\delta = \bigcup D'$ (where $\alpha + 1 := \alpha \cup \{\alpha\}$). Then δ is an ordinal as well, and $\forall \alpha \in D\, \alpha \in \delta$. Thus it follows that

$$\forall x \in b\, \exists \alpha \in \delta\, \exists e \in |\mathcal{A}|\, \exists z \in \mathrm{V}(\mathcal{A})_\alpha\ x = \langle e, z \rangle.$$

And hence, $b \subseteq \bigcup_{\alpha \in \delta} \mathcal{P}(|\mathcal{A}| \times \mathrm{V}(\mathcal{A})_\alpha$, so that $b \in \mathrm{V}(\mathcal{A})_\delta \subseteq \mathrm{V}(\mathcal{A})$. $\square$

4 Defining realizability

Having shown that the class $\mathrm{V}(\mathcal{A})$ can be formalized in **CZF**, we now proceed to define a notion of extensional realizability over $\mathrm{V}(\mathcal{A})$, i.e., $e \Vdash \phi$ for $e \in |\mathcal{A}|$ and sentences ϕ with parameters in $\mathrm{V}(\mathcal{A})$. Except for the special treatment of bounded quantifiers, this definition is due to McCarty [16]. For $e \in |\mathcal{A}|$ we shall write $(e)_0$ and $(e)_1$ rather than $\mathbf{p}_0 e$ and $\mathbf{p}_1 e$, respectively.

Definition: 4.1 Bounded quantifiers will be treated as quantifiers in their own right, i.e., bounded and unbounded quantifiers are treated as syntactically different kinds of quantifiers. Let $a, b \in \mathrm{V}(\mathcal{A})$ and $e \in |\mathcal{A}|$.

$$\begin{array}{rcl}
e \Vdash a \in b & \text{iff} & \exists c\, [(\langle (e)_0, c \rangle \in b \ \wedge\ (e)_1 \Vdash a = c] \\
e \Vdash a = b & \text{iff} & \forall f, d\, [(\langle f, d \rangle \in a \ \rightarrow\ (e)_0 f \Vdash d \in b) \ \wedge\ (\langle f, d \rangle \in b \ \rightarrow\ (e)_1 f \Vdash d \in a))] \\
e \Vdash \phi \wedge \psi & \text{iff} & (e)_0 \Vdash \phi \ \wedge\ (e)_1 \Vdash \psi \\
e \Vdash \phi \vee \psi & \text{iff} & [(e)_0 = \mathbf{0} \ \wedge\ (e)_1 \Vdash \phi] \ \vee\ [(e)_0 = \mathbf{1} \ \wedge\ (e)_1 \Vdash \psi] \\
e \Vdash \neg \phi & \text{iff} & \forall f \in |\mathcal{A}|\ \neg f \Vdash \phi \\
e \Vdash \phi \rightarrow \psi & \text{iff} & \forall f \in |\mathcal{A}|\, [f \Vdash \phi \ \rightarrow\ ef \Vdash \psi] \\
e \Vdash \forall x \in a\, \phi & \text{iff} & \forall \langle f, c \rangle \in a\ ef \Vdash \phi[x/c] \\
e \Vdash \exists x \in a\, \phi & \text{iff} & \exists c\, (\langle (e)_0, c \rangle \in a \ \wedge\ (e)_1 \Vdash \phi[x/c]) \\
e \Vdash \forall x \phi & \text{iff} & \forall c \in \mathrm{V}(\mathcal{A})\ e \Vdash \phi[x/a] \\
e \Vdash \exists x \phi & \text{iff} & \exists c \in \mathrm{V}(\mathcal{A})\ e \Vdash \phi[x/a]
\end{array}$$

Notice that $e \Vdash u \in v$ and $e \Vdash u = v$ can be defined for arbitrary sets u, v, viz., not just for $u, v \in \mathrm{V}(\mathcal{A})$. The definitions of $e \Vdash u \in v$ and $e \Vdash u = v$ fall under the scope of definitions by transfinite recursion. More precisely, the functions

$$\begin{aligned} F_{\in}(u,v) &= \{e \in |\mathcal{A}| \,:\, e \Vdash u \in v\} \\ G_{=}(u,v) &= \{e \in |\mathcal{A}| \,:\, e \Vdash u = v\} \end{aligned}$$

can be defined (simultaneously) on $\mathrm{V} \times \mathrm{V}$ by recursion on the relation

$$\langle c, d\rangle \lhd \langle a, b\rangle \quad \text{iff} \quad (c = a \,\wedge\, d \in \mathbf{TC}(b)) \,\vee\, (d = b \,\wedge\, c \in \mathbf{TC}(a)).$$

4.1 The soundness theorem for intuitionistic predicate logic with equality

Except for the extra considerations concerning bounded quantifiers, the proofs of 4.2 and 4.3 are almost the same for **CZF** as the corresponding proofs for **IZF** given in [16].

Lemma: 4.2 *There are* $\mathbf{i_r}, \mathbf{i_s}, \mathbf{i_t}, \mathbf{i_0}, \mathbf{i_1} \in |\mathcal{A}|$ *such that for all* $a, b, c \in \mathrm{V}(\mathcal{A})$,

1. $\mathbf{i}_r \Vdash a = a$.
2. $\mathbf{i}_s \Vdash a = b \rightarrow b = a$.
3. $\mathbf{i}_t \Vdash (a = b \wedge b = c) \;\rightarrow\; a = c$.
4. $\mathbf{i}_0 \Vdash (a = b \wedge b \in c) \;\rightarrow\; a \in c$.
5. $\mathbf{i}_1 \Vdash (a = b \wedge c \in a) \;\rightarrow\; c \in b$.

Moreover, for each formula $\varphi(v, u_1, \ldots, u_r)$ *of* **CZF** *all of whose free variables are among* $v, u_1, \ldots, u_r$ *there exists* $\mathbf{i}_\varphi \in |\mathcal{A}|$ *such that for all* $a, b, c_1, \ldots, c_r \in \mathrm{V}(\mathcal{A})$,

$$\mathbf{i}_\varphi \Vdash \varphi(a, \vec{c}) \,\wedge\, a = b \;\rightarrow\; \varphi(b, \vec{c}),$$

where $\vec{c} = c_1, \ldots, c_r$.

Proof: Realizers for the universal closures of the above formulas can be taken from [16], chapter 2, sections 5 and 6. Thus the above assertions follow from the "genericity" of realizers of universal statements, i.e.,

$$e \Vdash \forall v \psi(v) \;\text{ iff }\; \forall a \; e \Vdash \psi(a).$$

$\square$

Theorem: 4.3 *Let* $\mathcal{D}$ *be a proof in intuitionistic predicate logic with equality of a formula* $\varphi(u_1, \ldots, u_r)$ *of* **CZF** *all of whose free variables are among* $u_1, \ldots, u_r$. *Then there is* $e_{\mathcal{D}} \in |\mathcal{A}|$ *such that* **CZF** *proves*

$$e_{\mathcal{D}} \Vdash \forall u_1 \ldots \forall u_r \, \varphi(u_1, \ldots, u_r).$$

Proof: With the exception of the logical principles

$$\forall u \in a\, \varphi(u) \;\leftrightarrow\; \forall u\, [u \in a \rightarrow \varphi(u)], \tag{4}$$

$$\exists u \in a\, \varphi(u) \;\leftrightarrow\; \exists u\, [u \in a \;\wedge\; \varphi(u)], \tag{5}$$

which relate bounded to unbounded quantifiers, the proof is literally the same as in [16], chapter 2, sections 5 and 6. Let $a \in \mathrm{V}(\mathcal{A})$ and φ be a formula with parameters in $\mathrm{V}(\mathcal{A})$. We find a realizer for the formula of (4) as follows:

$$\begin{array}{rl}
 & e \Vdash \forall v\, [v \in a \;\rightarrow\; \varphi(v)] \\
\text{iff} & \forall x \in \mathrm{V}(\mathcal{A})\, \forall f \in |\mathcal{A}|\, [f \Vdash x \in a \quad \text{then} \quad e \Vdash \varphi(x)] \\
\text{iff} & \forall x \in \mathrm{V}(\mathcal{A})\, \forall f \in |\mathcal{A}|\, [\exists c\, (\langle (f)_0, c\rangle \in a \;\wedge\; (f)_1 \Vdash x = c) \quad \text{then} \quad ef \Vdash \varphi(x)] \\
\text{then} & \forall c \forall f \in |\mathcal{A}|\, [(\langle (f)_0, c\rangle \in a \;\wedge\; (f)_1 \Vdash c = c) \quad \text{then} \quad ef \Vdash \varphi(c)] \\
\text{then} & \forall\, \langle g, c\rangle \in a \;\; e\, (\mathbf{p} g \mathbf{i}_r) \Vdash \varphi(c)] \\
\text{then} & \lambda g.e\, (\mathbf{p} g \mathbf{i}_r) \Vdash \forall v \in a\; \varphi(u).
\end{array}$$

Conversely, we have

$$\begin{array}{rl}
 & e \Vdash \forall v \in a\; \varphi(u) \\
\text{iff} & \forall\, \langle f, c\rangle \in a \;\; ef \Vdash \varphi(c) \\
\text{then} & \forall x \in \mathrm{V}(\mathcal{A})\, \forall g \in |\mathcal{A}|\, [\exists c\, (\langle (g)_0, c\rangle \in a \;\wedge\; (g)_1 \Vdash x = c) \quad \text{then} \quad \mathbf{i}_\varphi(\mathbf{p}(g)_1(e\,(g)_0)) \Vdash \varphi(x)] \\
\text{then} & \lambda g.\mathbf{i}_\varphi(\mathbf{p}(g)_1(e\,(g)_0)) \Vdash \forall u[u \in v \rightarrow \varphi(u)].
\end{array}$$

The constants $\mathbf{i}_r, \mathbf{i}_\varphi$ are from Lemma 4.2. Letting $\mathbf{m}$ be

$$\mathbf{p}(\lambda e.\lambda g.e\, (\mathbf{p} g \mathbf{i}_r))(\lambda e.\lambda g.\mathbf{i}_\varphi(\mathbf{p}(g)_1(e\,(g)_0))),$$

we get

$$\mathbf{m} \Vdash \forall \vec{w} \forall v (\forall u \in v\, \varphi(u) \;\leftrightarrow\; \forall u\, [u \in v \rightarrow \varphi(u)]),$$

where $\forall \vec{w}$ quantifies over the remaining free variables of φ.

Similarly one finds $\bar{\mathbf{m}}$ such that

$$\bar{\mathbf{m}} \Vdash \forall \vec{w} \forall v (\exists u \in v\, \varphi(u) \;\leftrightarrow\; \exists u\, [u \in v \;\wedge\; \varphi(u)]).$$

□

4.2 Realizability for bounded formulae

In the following we shall often have occasion to employ the fact that for a bounded formula $\varphi(v)$ with parameters from $\mathrm{V}(\mathcal{A})$ and $x \subseteq \mathrm{V}(\mathcal{A})$,

$$\{\langle e, c\rangle :\; e \in |\mathcal{A}| \;\wedge\; c \in x \;\wedge\; e \Vdash \varphi(c)\}$$

is a set. To prove this we shall consider an extended class of formulae.

Definition: 4.4 The *extended bounded formulae* are the smallest class of formulas containing the formulae of the form $x \in y$, $x = y$, $e \Vdash x \in y$, $e \Vdash x = y$, which is closed under $\wedge, \vee, \neg, \rightarrow$ and bounded quantification.

Lemma: 4.5 (**CZF**) *Separation holds for extended bounded formulae, i.e., for every extended bounded formula $\varphi(v)$ and set x, $\{v \in x : \varphi(v)\}$ is a set.*

Proof: Since $F_{\in}$ and $G_{=}$ are provably total functions of **CZF**, formulas of the form $e \Vdash x \in y$ and $e \Vdash x = y$ can be treated in the context of **CZF** as though they were atomic symbols of the language. This follows from [20], Proposition 2.4 or [4], Proposition 11.12.$\square$

Lemma: 4.6 (**CZF**) *Let $\varphi(v, u_1, \ldots, u_r)$ be a bounded formula of* **CZF** *all of whose free variables are among $u_1, \ldots, u_r$. Then there there is an extended bounded formula $\widetilde{\varphi}(v, u_1, \ldots, u_r)$ and $f_\varphi \in |\mathcal{A}|$ such that for all $a_1, \ldots, a_r \in V(\mathcal{A})$ and $e \in |\mathcal{A}|$,*

$$e \Vdash \varphi(\vec{a}) \quad \textit{iff} \quad \widetilde{\varphi}(f_\varphi e, \vec{a}).$$

Proof: We proceed by induction on the generation of φ. For an atomic formula φ, the assertion follows with $\widetilde{\varphi} \equiv \varphi$ and f_φ being an index for the identity function of $\mathcal{A}$. The assertion easily follows from the respective inductive assumptions if φ is of the form $\varphi_0 \wedge \varphi_1$ or $\varphi_0 \vee \varphi_1$.

Now suppose φ is of the form $\forall x \in w\, \psi(x, \vec{u}, w)$. Inductively we then have for all $b, c, \vec{a} \in \mathrm{V}(\mathcal{A})$ and $e' \in |\mathcal{A}|$,

$$e' \Vdash \psi(b, \vec{a}, c) \quad \text{iff} \quad \widetilde{\psi}(f_\psi e', b, \vec{a}, c)$$

for some extended bounded formula $\widetilde{\psi}$. Hence, by the definition of realizability for bounded formulae, we can readily construct the desired extended formula $\widetilde{\varphi}$ from $\widetilde{\psi}$.

The case of a bounded existential quantifier is similar to the preceding case. $\square$

Corollary: 4.7 (**CZF**) *Let $\varphi(v)$ be a bounded formula with parameters from $\mathrm{V}(\mathcal{A})$ and $x \subseteq \mathrm{V}(\mathcal{A})$. Then*

$$\{\langle e, c\rangle : \; e \in |\mathcal{A}| \;\wedge\; c \in x \;\wedge\; e \Vdash \varphi(c)\}$$

is a set. Moreover, this set belongs to $\mathrm{V}(\mathcal{A})$.

Proof: The above class is a set by the previous two lemmas. That the set is also an element of $\mathrm{V}(\mathcal{A})$ follows from Lemma 3.5. $\square$

5 The soundness theorem for CZF

The soundness of extensional realizability for **IZF** was shown in [16]. The proofs for the realizability of Extensionality, Pair, Infinity, and Set Induction carry over to the context of **CZF**. Union needs a little adjustment to avoid an unnecessary appeal to unbounded Separation. To establish realizability of Bounded Separation we use Separation for extended bounded formulae. Strong Collection and in particular Subset Collection are not axioms of **IZF** and therefore require new proofs.

Theorem: 5.1 *For every axiom θ of* **CZF**, *there exists a closed application term t such that*

$$\mathbf{CZF} \vdash \ (t \Vdash \theta).$$

Proof: We treat the axioms one after the other.

(**Extensionality**): One easily checks that with

$$e \;\; = \;\; \lambda y.\mathbf{p}(\lambda x.\mathbf{p_0}y(\mathbf{p}x\mathbf{i}_r))(\lambda x.\mathbf{p_1}y(\mathbf{p}x\mathbf{i}_r)),$$

$e \Vdash \forall a\, \forall b\, [\forall x\, (x \in a \leftrightarrow x \in b) \; \rightarrow \; a = b]$.

(**Pair**): We need to guarantee the existence of an $e \in |\mathcal{A}|$ such that

$$\forall a, b \in \mathrm{V}(\mathcal{A}) \; e \Vdash a \in c \, \wedge \, b \in c \tag{6}$$

for some $c \in \mathrm{V}(\mathcal{A})$. Set $e = \mathbf{p}(\mathbf{p}0\mathbf{i}_r)(\mathbf{p}0\mathbf{i}_r)$ and let $c = \{\langle 0, a\rangle, \langle 0, b\rangle\}$. By Lemma 3.5, $c \in \mathrm{V}(\mathcal{A})$. One easily verifies that (6) holds for the specified e and c.

(**Union**): For each $a \in \mathrm{V}(\mathcal{A})$, put

$$Un^{\mathcal{A}}(a) \;\; = \;\; \{\langle h, y\rangle : \; h \in \mathcal{A} \; \wedge \; \exists \langle f, x\rangle \in a \; \langle h, y\rangle \in x\},$$

so that by Lemma 3.5, $Un^{\mathcal{A}}(a) \in \mathrm{V}(\mathcal{A})$.

Now assume $\langle f, x\rangle \in a$ and $\langle h, y\rangle \in x$. Then $\langle h, y\rangle \in Un^{\mathcal{A}}(a)$ and $\mathbf{p}h\mathbf{i}_r \Vdash y \in Un^{\mathcal{A}}(a)$, and hence, letting $e = \lambda u.\lambda v.\mathbf{p}v\mathbf{i}_r$, we have $e \Vdash \forall a\, \exists w\, \forall x \in a\, \forall y \in x\, y \in w$.

(**Bounded Separation**): Let $\varphi(x)$ be a bounded formula with parameters in $\mathrm{V}(\mathcal{A})$. This time we need to find $e, e' \in |A|$ such that for all $a \in \mathrm{V}(\mathcal{A})$ there exists a $b \in \mathrm{V}(\mathcal{A})$ such that

$$(e \Vdash \forall x \in b\, [x \in a \, \wedge \, \varphi(x)]) \; \wedge \; (e' \Vdash \forall x \in a[\varphi(x) \rightarrow x \in b]). \tag{7}$$

For $a \in \mathrm{V}(\mathcal{A})$, define

$$Sep^{\mathcal{A}}(a, \varphi) \;\; = \;\; \{\langle \mathbf{p}fg, x\rangle : \; f, g \in \mathcal{A} \; \wedge \; \langle g, x\rangle \in a \; \wedge \; f \Vdash \varphi(x)\}.$$

By Corollary 4.7, $Sep^{\mathcal{A}}(a, \varphi) \in \mathrm{V}(\mathcal{A})$. Put $b = Sep^{\mathcal{A}}(a, \varphi)$.

To verify (7), first assume $\langle \mathbf{p}fg, x\rangle \in b$. Then, by definition of b, $\langle g, x\rangle \in a$ and $f \Vdash \varphi(x)$, so that with

$$e = \mathbf{p}(\mathbf{p}(\lambda u.(u)_1)\mathbf{i}_r)(\lambda u.(u)_0),$$

$e \Vdash \forall x \in b\,[x \in a \wedge \varphi(x)]$.

Now assume $\langle g, x\rangle \in a$ and $f \Vdash \varphi(x)$. Then $\langle \mathbf{p}fg, x\rangle \in b$. Hence with $e' = \lambda u.\lambda v.\mathbf{p}(\mathbf{p}vu)\mathbf{i}_r$ we get $e' \Vdash \forall x \in a[\varphi(x) \rightarrow x \in b]$.

(**Infinity**): The most obvious candidate to represent ω in $\mathrm{V}(\mathcal{A})$ is $\overline{\omega}$, which is given via an injection of ω into $\mathrm{V}(\mathcal{A})$. Recall that $\mathbf{0}$ denotes the zero of $\mathcal{A}$, $\mathbf{1} = \mathbf{s}_N\,\mathbf{0}$ and that 0 (the empty set) is the least element of ω. Set $\underline{0} = \mathbf{0}$ and for $n \in \omega$, let $\underline{n+1} = \mathbf{s}_N\,\underline{n}$ and set $\overline{n} = \{\langle \underline{m}, \overline{m}\rangle : m \in n\}$. Then, we take

$$\overline{\omega} \quad = \quad \{\langle \underline{n}, \overline{n}\rangle : n \in \omega\}.$$

Clearly, by Lemma 3.5, $\overline{\omega} \in \mathrm{V}(\mathcal{A})$. Note also that $N(\underline{n})$ holds for all $n \in \omega$. Moreover, the applicative axioms imply that if $n, m \in \omega$ and $n \neq m$, then $\underline{n} \neq \underline{m}$.

In order to show realizability of the Infinity axiom, we first have to write it out in full detail. Let $\bot_v$ be the formula $\forall u \in v\ \neg u = u$ and let $SC(u,v)$ be the formula $\forall y \in v\,[y = u \ \vee\ y \in u] \ \wedge\ [u \in v \ \wedge\ \forall y \in u\, y \in v]$. Then Infinity amounts to the sentence

$$\exists x\,(\forall v \in x\,[\bot_v \ \vee \exists u \in x\, SC(u,v)] \ \wedge\ \forall v[(\bot_v \ \vee \exists u \in x\, SC(u,v)) \ \rightarrow\ v \in x]). \tag{8}$$

Suppose $\langle f, c\rangle \in \overline{\omega}$. Then $f = \underline{n}$ and $c = \overline{n}$ for some $n \in \omega$. If $n = 0$ then $\overline{n} = 0$ and therefore $\mathbf{0} \Vdash \bot_{\overline{n}}$. Otherwise we have $n = k+1$ for some $k \in \omega$. If $\langle \underline{m}, \overline{m}\rangle \in \overline{n}$ then $m = k$ or $m \in k$, so that $\mathbf{i}_r \Vdash \overline{m} = \overline{k}$ or $\mathbf{p}\underline{m}\mathbf{i}_r \Vdash \overline{m} \in \overline{k}$, and whence $\mathbf{d}(\mathbf{p0i}_r)(\mathbf{p1}(\mathbf{p}\underline{m}\mathbf{i}_r))\underline{m}\,\underline{k} \Vdash (\overline{m} = \overline{k} \ \vee\ \overline{m} \in \overline{k})$. As a result of the foregoing we have $\ell(\underline{k}) \Vdash \forall y \in \overline{n}\,(y = \overline{k} \ \vee\ y \in \overline{k})$, where $\ell(\underline{k}) := \lambda z.\mathbf{d}(\mathbf{p0i}_r)(\mathbf{p1}(\mathbf{p}z\mathbf{i}_r))z\,\underline{k}$. Note both that $\mathbf{p}\underline{k}\mathbf{i}_r \Vdash \overline{k} \in \overline{n}$ and $\lambda z.\mathbf{p}z\mathbf{i}_r \Vdash \forall y \in \overline{k}\, y \in \overline{n}$, and hence $\wp(\underline{k}) \vdash \overline{k} \in \overline{n} \ \wedge\ \forall y \in \overline{k}\, y \in \overline{n}$, where $\wp(\underline{k}) := \mathbf{p}(\mathbf{p}\underline{k}\mathbf{i}_r)(\lambda z.\mathbf{p}z\mathbf{i}_r)$. Also note that $\underline{k} = \mathbf{p}_N\,\underline{n}$. With

$$t(\underline{n}) \quad := \quad \mathbf{p}(\mathbf{p}_N\underline{n})(\mathbf{p}(\ell(\mathbf{p}_N\,\underline{n}))(\wp(\mathbf{p}_N\,\underline{n})))$$

we thus obtain $t(\underline{n}) \Vdash \exists u \in \overline{\omega}\, SC(u, \overline{n})$. In conclusion, as $n = 0$ or $n = k+1$ for some $k \in \omega$ and $\underline{n} = f$ and $\overline{n} = c$ we arrive at $\mathbf{d}(\mathbf{p00})(\mathbf{p1}t(f))f\mathbf{0} \Vdash [\bot_c \ \vee \exists u \in \overline{\omega}\, SC(u,c)]$. Hence we have

$$\mathbf{q}^* \Vdash \forall v \in \overline{\omega}\,[\bot_v \ \vee \exists u \in \overline{\omega}\, SC(u,v)] \tag{9}$$

where $\mathbf{q}^* := \lambda f.\mathbf{d}(\mathbf{p00})(\mathbf{p1}t(f))f\mathbf{0}$.

Conversely assume $a \in \mathrm{V}(\mathcal{A})$ and

$$e \ \ \Vdash \ \ \bot_a \vee \exists u \in \overline{\omega}\, SC(u,a). \tag{10}$$

Then either $(e)_0 = \mathbf{0}$ and $(e)_1 \Vdash \bot_a$ or $(e)_0 = \mathbf{1}$ and $(e)_1 \Vdash \exists u \in \overline{\omega}\, SC(u,a)$.

The first case scenario yields $a = \emptyset = 0$. To see this assume $\langle f, c\rangle \in a$. Then $(e)_1 f \Vdash \neg c = c$, which means that $\forall g \in |\mathcal{A}|\ \neg g \Vdash c = c$. However, as $\mathbf{i}_r \Vdash c = c$ this is absurd, showing $a = 0$. The latter yields $\mathbf{i}_r \Vdash \overline{0} = a$ and thus

$$\mathbf{p}(e)_0\mathbf{i}_r \Vdash a \in \overline{\omega}. \tag{11}$$

The second scenario entails that $((e)_1)_0 = \underline{n}$ for some $n \in \omega$ as well as $((e)_1)_1 \Vdash SC(\overline{n}, a)$. Therefore we can conclude that $t_1 \Vdash \forall y \in a\,(y = \overline{n} \vee y \in \overline{n})$, $t_2 \Vdash \overline{n} \in a$, and $t_3 \Vdash \forall y \in \overline{n}\, y \in a$ with $s := ((e)_1)_1$, $t_1 := (s)_0$, $t_2 := ((s)_1)_0$ and $t_3 := ((s)_1)_1$. Our first aim is to construct a closed application term $\mathbf{q}^{\#}$ such that $\mathbf{q}^{\#} \Vdash a = \overline{n+1}$. To this end assume first that $\langle f, c\rangle \in a$. Then $t_1 f \Vdash c = \overline{n} \vee c \in \overline{n}$ and $(t_1 f)_0 = \mathbf{0}$ or $(t_1 f)_0 = \mathbf{1}$. From $(t_1 f)_0 = \mathbf{0}$ we obtain $(t_1 f)_1 \Vdash c = \overline{n}$, and hence $\mathbf{p}\underline{n}(t_1 f)_1 \Vdash c \in \overline{n+1}$. If, on the other hand, $(t_1 f)_0 = \mathbf{1}$, we conclude that $(t_1 f)_1 \Vdash c \in \overline{n}$, which entails that $((t_1 f)_1)_0 = \underline{k}$ and $((t_1 f)_1)_1 \Vdash c = \overline{k}$ for some $k \in n$, and hence $\mathbf{p} r_0 r_1 \Vdash c \in \overline{n+1}$ where $r_i := ((t_1 f)_1)_i$. To summarize, we have

$$\langle f, c\rangle \in a \quad \to \quad q_1(f) \Vdash c \in \overline{n+1}, \tag{12}$$

where $q_1(f) := \mathbf{d}(\mathbf{p}\underline{n}(t_1 f)_1)(\mathbf{p} r_0 r_1)(t_1 f)_0 \mathbf{0}$.

Next assume that $\langle f, c\rangle \in \overline{n+1}$. Thus $f = \underline{k}$ and $c = \overline{k}$ for some $k \in n+1$. We then have $k = n \vee k \in n$. $k = n$ yields $t_2 \Vdash c \in a$, while $k \in n$ yields $t_3 \underline{k} \Vdash \overline{k} \in a$, so that $t_3 \underline{k} \Vdash c \in a$. Thus, since $f = \underline{k}$ we get $q_2(f) \Vdash c \in a$ with $q_2(f) := \mathbf{d} t_2 (t_3 f) f \underline{n}$. In conclusion,

$$\langle f, c\rangle \in \overline{n+1} \quad \to \quad q_2(f) \Vdash c \in a. \tag{13}$$

With $\mathbf{q}^{\#} := \mathbf{p}(\lambda f.q_1(f))(\lambda f.q_2(f))$, (12) and (13) entail that $\mathbf{q}^{\#} \Vdash a = \overline{n+1}$, and thus $\mathbf{p}\underline{n+1}\mathbf{q}^{\#} \Vdash a \in \overline{\omega}$.

The upshot of the foregoing is that from (10) we have concluded that (11) holds if $(e)_0 = \mathbf{0}$ and that (13) holds if $(e)_0 = \mathbf{1}$. Also note that $(e)_0 = \mathbf{1}$ entails $\underline{n+1} = \mathbf{s}_N\, \underline{n} = \mathbf{s}_N\, ((e)_1)_0$. Thus we arrive at $\ell^{**}(e) \Vdash a \in \overline{\omega}$ with $\ell^{**}(e) := \mathbf{d}(\mathbf{p}(e)_0 \mathbf{i}_r)(\mathbf{p}(\mathbf{s}_N\, ((e)_1)_0)\mathbf{q}^{\#})(e)_0 \mathbf{0}$. Using lambda-abstraction on e, it follows that

$$\lambda e.\ell^{**}(e) \quad \Vdash \quad \forall v[(\bot_v \vee \exists u \in \overline{\omega}\, SC(u, v)) \to v \in \overline{\omega}]. \tag{14}$$

Finally, (9) and (14) yield that $\mathbf{p}\mathbf{q}^{*}(\lambda e.\ell^{**}(e))$ provides a realizer for the Infinity axiom.

(**Set Induction**): Assume that for all $a \in \mathrm{V}(\mathcal{A})$, $g \Vdash (\forall y \in a \varphi(y)) \to \varphi(a)$. We would like to construct an $e \in |\mathcal{A}|$ such that for all $b \in \mathrm{V}(\mathcal{A})$, $eg \Vdash \varphi(b)$. To this end, suppose that $a \in \mathrm{V}(\mathcal{A})_\alpha$ and that we have found an $e \in |\mathcal{A}|$ such that for all $b \in \bigcup_{\beta \in \alpha} \mathrm{V}(\mathcal{A})_\beta$, $e \Vdash \varphi(b)$. Now, if $\langle h, b\rangle \in a$, then $b \in \bigcup_{\beta\in\alpha} \mathrm{V}(\mathcal{A})_\beta$, and hence $e \Vdash \varphi(b)$, so that

$$\begin{aligned} &\lambda u.\mathbf{k} e u \Vdash \forall y \in a \varphi(y) \text{ and} \\ &\qquad g(\lambda u.\mathbf{k} e u) \Vdash \varphi(a). \end{aligned}$$

By the recursion theorem for applicative structures (cf. [16], chap. 2, Corollary 2.7), there is an $f^* \in |\mathcal{A}|$ that fixes $e^* = g(\lambda u.\mathbf{k} e u)$, i.e., $f^* e^* = g(\lambda u.\mathbf{k}(f^* e^*) u)$. Then, with $e = \lambda g.f^* e^*$, induction on α yields

$$e \Vdash \forall a\, [(\forall y \in a \varphi(y)) \to \varphi(a)] \to \forall a\, \varphi(a).$$

(**Strong Collection**): Let $a \in \mathrm{V}(\mathcal{A})$ and assume that $g \Vdash \forall x \in a\, \exists y\, \varphi(x, y)$. Then, for $\langle u, x\rangle \in a$ there is a $y \in \mathrm{V}(\mathcal{A})$ such that $gu \Vdash \varphi(x, y)$. By invoking Strong Collection in the background universe, there is a set D such that

$$\forall\langle u,x\rangle\in a\,\exists y\in \mathrm{V}(\mathcal{A})\,[\langle \mathbf{p}(gu)u,y\rangle\in D\ \wedge\ gu\Vdash\varphi(x,y)],\qquad\text{and}\tag{15}$$

$$\forall z\in D\,\exists\langle u,x\rangle\in a\,\exists y\in \mathrm{V}(\mathcal{A})\,[z=\langle \mathbf{p}(gu)u,y\rangle\ \wedge\ gu\Vdash\varphi(x,y)].\tag{16}$$

In particular, $D\subseteq|\mathcal{A}|\times\mathrm{V}(\mathcal{A})$, so that by Lemma 3.5, $D\in\mathrm{V}(\mathcal{A})$. We need to construct $e,e'\in\mathrm{V}(\mathcal{A})$ from g such that

$$e\Vdash\forall x\in a\,\exists y\in D\,\varphi(x,y),\tag{17}$$

$$e'\Vdash\forall y\in D\,\exists x\in a\,\varphi(x,y).\tag{18}$$

For (17), let $\langle u,x\rangle\in a$. Then there exists a y such that $\langle\mathbf{p}(gu)u,y\rangle\in D$ and $gu\Vdash\varphi(x,y)$, and hence $\mathbf{p}(\mathbf{p}(gu)u)(gu)\Vdash\exists y\in D\,\varphi(x,y)$; so that, with $e=\lambda u.\mathbf{p}(\mathbf{p}(gu)u)(gu)$, (17) obtains.

To show (18), let $\langle v,y\rangle\in D$. Then, by (16), $v=\mathbf{p}(gu)u$ for some $u\in\mathrm{V}(\mathcal{A})$ and there exists an x such that $\langle u,x\rangle\in a\ \wedge\ gu\Vdash\varphi(x,y)$. Hence $\mathbf{p}(v)_1(v)_0\Vdash\exists x\in a\,\varphi(x,y)$, so that with $e'=\lambda v.\mathbf{p}(v)_1(v)_0$, (18) obtains.

(**Subset Collection**): Let $a,b\in\mathrm{V}(\mathcal{A})$ and $\varphi(x,u,y)$ be a formula with parameters in $\mathrm{V}(\mathcal{A})$. We would like to find a realizer $\mathbf{r}$ such that

$$\mathbf{r}\Vdash\exists q\forall u[\forall x\in a\,\exists y\in b\,\varphi(x,y,u)\rightarrow\exists v\in q\,\varphi'(a,v,u)],\tag{19}$$

where $\varphi'(a,v,u)$ abbreviates the formula

$$\forall x\in a\,\exists y\in v\,\varphi(x,y,u)\ \wedge\ \forall y\in v\,\exists x\in a\,\varphi(x,y,u).$$

Set

$$b^*\ =\ \{\langle\mathbf{p}ef,d\rangle\ :\ e,f\in|\mathcal{A}|\ \wedge\ ef\downarrow\ \wedge\langle(ef)_0,d\rangle\in b\}.$$

Note that b^* is a set. Further, let $\psi(e,f,c,u,z)$ be the formula

$$u\in\mathrm{V}(\mathcal{A})\ \wedge\ e,f\in|\mathcal{A}|\ \wedge\ ef\downarrow\ \wedge\exists d\,[\langle\mathbf{p}ef,d\rangle=z\ \wedge\ \langle(ef)_0,d\rangle\in b\ \wedge\ (ef)_1\Vdash\varphi(c,d,u)].$$

By invoking Subset Collection there exists a set D such that

$$\begin{aligned}&\forall u\forall e\,[\forall\,\langle f,c\rangle\in a\,\exists z\in b^*\psi(e,f,c,u,z)\ \rightarrow\\&\qquad\exists w\in D\,(\forall\,\langle f,c\rangle\in a\,\exists z\in w\,\psi(e,f,c,u,z)\ \wedge\ \forall z\in w\,\exists\,\langle f,c\rangle\in a\,\psi(e,f,c,u,z))].\end{aligned}$$

Now set

$$D^*\ :=\ \{w\cap b^*\ :\ w\in D\}.$$

Then $D^*\subseteq\mathrm{V}(\mathcal{A})$, and thus

$$E\ :=\ \{\langle 0,v\rangle\ :\ v\in D^*\}$$

is an element of $\mathrm{V}(\mathcal{A})$.

Let $e\in|\mathcal{A}|$ and $u\in\mathrm{V}(\mathcal{A})$ satisfy

$$e\Vdash\forall x\in a\,\exists y\in b\,\varphi(x,y,u).\tag{20}$$

Then

$$\forall \langle f,c\rangle \in a\, \exists d\, [\langle (ef)_0, d\rangle \in b \;\wedge\; (ef)_1 \Vdash \varphi(c,d,u)].$$

Therefore we get $\forall \langle f,c\rangle \in a\, \exists z \in b^*\, \psi(e,f,c,u,z)$. Thus there exists $v \in D^*$ such that

$$\begin{aligned} &\forall \langle f,c\rangle \in a\, \exists z \in v\, \psi(e,f,c,u,z) \;\wedge \\ &\quad \forall z \in v\, \exists \langle f,c\rangle \in a\, \psi(e,f,c,u,z). \end{aligned}$$

The latter implies the following two assertions:

$$\begin{gathered} \forall \langle f,c\rangle \in a\, \exists d\, [\langle \mathbf{p}ef, d\rangle \in v \;\wedge\; \langle (ef)_0, d\rangle \in b \;\wedge\; (ef)_1 \Vdash \varphi(c,d,u)], \\ \forall z \in v\, \exists \langle f,c\rangle \in a\, \exists d\, [z = \langle \mathbf{p}ef, d\rangle \;\wedge\; \langle (ef)_0, d\rangle \in b \;\wedge\; (ef)_1 \Vdash \varphi(c,d,u)]. \end{gathered}$$

With

$$\begin{aligned} \mathbf{m}_0 &:= \lambda f.\mathbf{p}(\mathbf{p}ef)(ef)_1, \\ \mathbf{m}_1 &:= \lambda g.\mathbf{p}((g)_0(g)_1)_0((g)_0(g)_1)_1 \end{aligned}$$

we thus obtain

$$\begin{gathered} \mathbf{m}_0 \Vdash \forall x \in a\, \exists y \in v\, \varphi(x,y,u), \\ \mathbf{m}_1 \Vdash \forall y \in v\, \exists x \in a\, \varphi(x,y,u). \end{gathered}$$

As a result of the foregoing we have

$$\mathbf{p}\mathbf{m}_0\mathbf{m}_1 \Vdash \forall x \in a\, \exists y \in v\, \varphi(x,y,u) \;\wedge\; \forall y \in v\, \exists x \in a\, \varphi(x,y,u). \tag{21}$$

Let $\varphi'(a,v,u)$ stand for $\forall x \in a\, \exists y \in v\, \varphi(x,y,u) \;\wedge\; \forall y \in v\, \exists x \in a\, \varphi(x,y,u)$. Thus far we have shown that (20) implies (21). Consequently,

$$\lambda e.\mathbf{p}0(\mathbf{p}\mathbf{m}_0\mathbf{m}_1) \Vdash \forall x \in a\, \exists y \in b\, \varphi(x,y,u) \;\rightarrow\; \exists v \in E\, \varphi'(a,v,u),$$

and thus

$$\lambda e.\mathbf{p}0(\mathbf{p}\mathbf{m}_0\mathbf{m}_1) \Vdash \exists q\, \forall u\, [\forall x \in a\, \exists y \in b\, \varphi(x,y,u) \;\rightarrow\; \exists v \in q\, \varphi'(a,v,u)],$$

verifying Subset Collection. ☐

6 The soundness theorem for CZF + REA

Next we show that the regular extension axiom holds in $\mathrm{V}(\mathcal{A})$ if it holds in the background universe.

Lemma: 6.1 (**CZF**)

1. *If B is a regular set with $2 \in B$, then B is closed under unordered and ordered pairs, i.e., whenever $x, y \in B$, then $\{x,y\}, \langle x,y\rangle \in B$.*

2. If B is a regular set, then $B \cap \mathrm{V}(\mathcal{A})$ is a set.

Proof: (1): Let $x, y \in B$. Then $f : 2 \to B$, where $f(0) = x$, $f(1) = y$. Hence, by regularity of B, the range of f is in B, that is $\{x, y\} \in B$.

As $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$, $\langle x, y\rangle \in B$ follows from closure under unordered pairs.

(ii): To see this let $\kappa = \mathrm{rank}(B)$, where the function rank is defined by $\mathrm{rank}(x) := \bigcup\{\mathrm{rank}(y)+1 \,:\, y \in x\}$ with $z+1 := z \cup \{z\}$. One easily shows that for all sets x, $\mathrm{rank}(x)$ is an ordinal. Let Φ be the inductive definition with

$$\frac{x}{a}\,\Phi \quad \text{iff} \quad \forall u{\in}a\ (u \in |\mathcal{A}| \times x).$$

Invoking Lemma 3.4, let J be the class such that $\mathrm{V}(\mathcal{A}) = \bigcup_\alpha J^\alpha$, and for each α,

$$J^\alpha \;=\; \Gamma_\Phi(\bigcup_{\beta\in\alpha} J^\beta).$$

Moreover, define the operation Υ by

$$\Upsilon(X) \;:=\; \{u \in B \,:\, u \subseteq |\mathcal{A}| \times X\}$$

and by recursion on α set

$$\Upsilon^\alpha \;=\; \Upsilon(\bigcup_{\beta\in\alpha} \Upsilon^\beta).$$

Let

$$\Upsilon^{<\nu} := \bigcup_{\xi\in\nu} \Upsilon^\xi.$$

Then $\Upsilon^{<\kappa}$ is a set and $\Upsilon^{<\kappa} \subseteq \mathrm{V}(\mathcal{A})$. By induction on α we shall verify that

$$J^\alpha \cap B \subseteq \Upsilon^{<\kappa}. \tag{22}$$

Thus assume that for all $\beta \in \alpha$, $J^\beta \cap B \subseteq \Upsilon^{<\kappa}$. If $c \in J^\alpha \cap B$, then

$$c \subseteq |\mathcal{A}| \times \bigcup_{\beta\in\alpha}(J^\beta \cap B)$$

since B is transitive. Hence for every $x \in c$, there exists $e \in |\mathcal{A}|$ and $u \in \Upsilon^{<\kappa}$ such that $x = \langle e, u\rangle$. Thus, by definition of κ, for every $x \in c$ there exist $u, v \in B$, $e \in |\mathcal{A}|$, such that with $\eta = \mathrm{rank}(v)$, $u \in \Upsilon^\eta$ and $x = \langle e, u\rangle$. Using the regularity of B, there exists a set $d \in B$ such that for each $x \in c$ there exist $u \in B$, $v \in d$, $e \in |\mathcal{A}|$, such that with $\eta = \mathrm{rank}(v)$, $u \in \Upsilon^\eta$ and $x = \langle e, u\rangle$. As a result,

$$c \subseteq |\mathcal{A}| \times \Upsilon^{<\gamma},$$

where $\gamma = \mathrm{rank}(d)$. Thus $c \in \Upsilon^\gamma \subseteq \Upsilon^{<\kappa}$ as $\gamma \in \kappa$. Consequently, we have $J^\alpha \cap B \subseteq \Upsilon^{<\kappa}$.

The upshot of the above is that

$$B \cap \mathrm{V}(\mathcal{A}) \;=\; \Upsilon^{<\kappa}$$

and therefore $B \cap \mathrm{V}(\mathcal{A})$ is set. □

Theorem: 6.2 *For every axiom θ of* $\mathbf{CZF}+\mathbf{REA}$*, there exists a closed application term t such that*

$$\mathbf{CZF}+\mathbf{REA} \vdash\ (t \Vdash \theta).$$

Proof: In view of theorem 5.1, we need only find a realizer for the axiom **REA**. Let $a \in \mathrm{V}(\mathcal{A})$. By **REA** there exists a regular set B such that $a, 2, |\mathcal{A}| \in B$. Let

$$\begin{array}{rcl} A & := & B \cap \mathrm{V}(\mathcal{A}), \\ C & := & \{\langle 0, x\rangle\ :\ x \in A\}. \end{array}$$

By Lemma 6.1, A is a set; hence C is a set. Moreover, as $A \subseteq \mathrm{V}(\mathcal{A})$, it follows $C \in \mathrm{V}(\mathcal{A})$ and

$$\mathbf{p}0\mathbf{i}_r \Vdash a \in C. \tag{23}$$

Next we would like to find a realizer $\mathbf{q}$ such that

$$\mathbf{q} \Vdash \mathbf{Reg}(C). \tag{24}$$

To this end, suppose that $b \in \mathrm{V}(\mathcal{A})$, $f \in |\mathcal{A}|$, and $\varphi(x,y)$ is a formula with parameters in $\mathrm{V}(\mathcal{A})$ satisfying

$$f \Vdash b \in C\ \wedge\ \forall x \in b\, \exists y \in C\, \varphi(x,y). \tag{25}$$

Then there exists d such that

$$\langle f_{0,0}, d\rangle \in C\ \ \wedge\ \ f_{0,1} \Vdash b = d, \tag{26}$$

$$f_1 \Vdash \forall x \in b\, \exists y \in C\, \varphi(x,y), \tag{27}$$

where $f_{0,0} := ((f)_0)_0$, $f_{0,1} := ((f)_0)_1$, and $f_1 := (f)_1$. (26) and (27) yield

$$\mathbf{i}_\psi f_{0,1} f_1 \Vdash \forall x \in d\, \exists y \in C\, \varphi(x,y), \tag{28}$$

where $\mathbf{i}_\psi \Vdash \forall u \forall v[u = v \wedge \psi(u) \to \psi(v)]$, with $\psi(u)$ being $\forall x \in u\, \exists y \in C\, \varphi(x,y)$ (according to Lemma 4.2, $\mathbf{i}_\psi$ is independent of C and the parameters in φ).

Since B is closed under ordered pairs (Lemma 6.1) and $|\mathcal{A}| \in B$, from (28) we get

$$\forall x \forall e\, [\langle e, x\rangle \in d\ \to\ \exists z \in B\, \exists y(z = \langle e, y\rangle\ \wedge\ \langle (\tilde{f}e)_0, y\rangle \in C\ \wedge\ (\tilde{f}e)_1 \Vdash \varphi(x,y))], \tag{29}$$

where $\tilde{f} := \mathbf{i}_\psi f_{0,1} f_1$. Noting that $\langle v, y\rangle \in C$ entails $v = 0$, and utilizing the regularity of B, there exists $\mathbf{u} \in B$ such that

$$\forall x \forall e\, [\langle e, x\rangle \in d\ \to\ \exists z \in \mathbf{u}\, \exists y(z = \langle e, y\rangle\ \wedge\ \langle 0, y\rangle \in C\ \wedge\ (\tilde{f}e)_1 \Vdash \varphi(x,y))]; \tag{30}$$

$$\forall z \in \mathbf{u}\, \exists x, e\, [\langle e, x\rangle \in d\ \wedge\ \exists y(\langle 0, y\rangle \in C\ \wedge\ z = \langle e, y\rangle\ \wedge\ (\tilde{f}e)_1 \Vdash \varphi(x,y))]. \tag{31}$$

From (31) it follows that $\mathbf{u} \in A$, and thus $\langle 0, \mathbf{u}\rangle \in C$. So we get

$$\mathbf{p}0\mathbf{i}_r \Vdash \mathbf{u} \in C. \tag{32}$$

Letting $s(f) := \lambda e.\mathbf{p}e(\tilde{f}e)_1$, (30) and (31) yield

$$s(f) \Vdash \forall x \in d\,\exists y \in \mathbf{u}\,\varphi(x,y),$$
$$s(f) \Vdash \forall y \in \mathbf{u}\,\exists x \in d\,\varphi(x,y).$$

By invoking Lemma 4.2, from the latter we can effectively determine application terms $\tilde{s}(f)$ and $\hat{s}(f)$ such that

$$\tilde{s}(f) \Vdash \forall x \in b\,\exists y \in \mathbf{u}\,\varphi(x,y), \tag{33}$$
$$\hat{s}(f) \Vdash \forall y \in \mathbf{u}\,\exists x \in b\,\varphi(x,y). \tag{34}$$

Hence, with $\mathbf{q} := \lambda f.\mathbf{p}(\mathbf{p}0\mathbf{i}_r)(\mathbf{p}\tilde{s}(f)\hat{s}(f))$, (32), (33), and (34) entail

$$\begin{aligned}\mathbf{q} \Vdash \forall b(b \in C \,\wedge\, \forall x \in b\,\exists y \in C\,\varphi(x,y) \,\rightarrow \\ \exists u \in C\,[\forall x \in b\,\exists y \in u\,\varphi(x,y) \,\wedge\, \forall y \in u\,\exists x \in b\,\varphi(x,y)]).\end{aligned}$$

Choosing $\varphi(x,y)$ to be the formula $r \subseteq b \times C \,\wedge\, \langle x,y\rangle \in r$, we get

$$\mathbf{q} \Vdash \mathbf{Reg}(C).$$

Thus, in view of (23), we get

$$\mathbf{p}(\mathbf{p}0\mathbf{i}_r)\mathbf{q} \Vdash \forall a\,\exists C\,[a \in C \,\wedge\, \mathbf{Reg}(C)].$$

□

7 Adding Markov's Principle and Independence of Premisses

Markov's Principle (**MP**) is closely associated with the work of the school of Russian constructivists. The version of **MP** most appropriate to the set-theoretic context is the schema

$$\forall n \in \omega\,[\varphi(n) \vee \neg\varphi(n)] \,\rightarrow\, [\neg\neg\exists n \in \omega\,\varphi(n) \rightarrow \exists n \in \omega\varphi(n)].$$

The variant

$$\neg\neg\exists n \in \omega\,R(n) \rightarrow \exists n \in \omega R(n)],$$

with R being a primitive recursive predicate, will be denoted by $\mathbf{MP_{PR}}$. Obviously, $\mathbf{MP_{PR}}$ is implied by **MP**.

Another classically valid principle considered in connection with intuitionistic theories is the *Principle of Independence of Premisses*, **IP**, which is expressed by the schema

$$(\neg\theta \rightarrow \exists x\psi) \,\rightarrow\, \exists x(\neg\theta \rightarrow \psi),$$

where θ is assumed to be closed. A variant of **IP** is $\mathbf{IP}_\omega$:

$$(\neg\theta \rightarrow \exists n \in \omega\,\psi) \,\rightarrow\, \exists n \in \omega\,(\neg\theta \rightarrow \psi),$$

where θ is closed.

As has been shown by McCarty, both **MP** and **IP** are realized in $\mathrm{V}(Kl)$ if one assumes classical logic in the background theory (cf. [16], Theorems 11.3 and 11.5). In connection with **CZF** one is naturally led to ask whether these principles add any proof-theoretic strength to **CZF**?

Theorem: 7.1 *(i)* **CZF** *and* $\mathbf{CZF} + \mathbf{MP} + \mathbf{IP} + \mathbf{IP}_\omega$ *have the same proof-theoretic strength and the same provably recursive functions.*

(ii) $\mathbf{CZF} + \mathbf{REA}$ *and* $\mathbf{CZF} + \mathbf{REA} + \mathbf{MP} + \mathbf{IP}_\omega$ *have the same proof-theoretic strength and the same provably recursive functions.*

(i) also obtains if one adds the axioms **DC** *(Dependent Choices),* **PAx** *(Presentation Axiom), and* $\Pi\Sigma$-**AC** *(for definitions see [3]) to* $\mathbf{CZF} + \mathbf{MP} + \mathbf{IP} + \mathbf{IP}_\omega$.

Likewise, in (ii) one may add **DC**, **PAx**, *and* $\Pi\Sigma W$-**AC** *to* $\mathbf{CZF} + \mathbf{REA} + \mathbf{MP} + \mathbf{IP} + \mathbf{IP}_\omega$.

Proof: All of these results are actually corollaries of the interpretations of the systems $\mathbf{ML_1V}$ and $\mathbf{ML_{1W}V}$ of Martin-Löf type theory into the classical set theories **KP** and **KPi** of Kripke-Platek set theory, respectively, given in [20], Theorems 4.11 and 5.13. Combining these interpretations with Aczel's formulae-as-types interpretations of set theory into Martin-Löf type theory, one obtains formulas-as-classes interpretations of **CZF** and $\mathbf{CZF} + \mathbf{REA}$ in **KP** and **KPi**, respectively.

To be more precise, we shall focus on the interpretation of **CZF** in **KP** obtained in this way. The first step consists in simulating $\mathbf{ML_1V}$ in **KP** by interpreting a type A as a class of natural numbers $[\![A]\!]$ and the equality relation $=_A$ on A as a class $[\![=_A]\!]$ of pairs of natural numbers. Pivotally, if A, B are types that have been interpreted as classes $[\![A]\!]$ and $[\![B]\!]$, then the function type $A \to B$ is interpreted as the class of indices e of partial recursive functions satisfying $\forall x \in [\![A]\!]\ \{e\}(x) \in [\![B]\!]$ and $\forall (x, y) \in [\![=_A]\!]\ (\{e\}(x), \{e\}(y)) \in [\![=_B]\!]$.

Reasoning in **KP** one inductively define classes **U** and **V** such that $\mathbf{U}, \mathbf{V} \subseteq \omega$. **U** is the class of codes of small types while **V** serves as a universe for the interpretation of formulas of **CZF** (for the precise definitions of $x \in \mathbf{U}$ and $x \in \mathbf{V}$ we have to refer to [20], Definition 4.3: $\mathbf{U} = \{x \in \omega : \mathbb{U} \models x \quad \mathbf{set}\}$ and $\mathbf{V} = \{x \in \omega : \mathbb{V} \models x \quad \mathbf{set}\}$). Moreover, each $\alpha \in \mathbf{V}$ is of the form $\sup(n, e)$ such that $n, e \in \omega$, sup is a primitive recursive pairing function on ω, $n \in \mathbf{U}$ and e is an index of a partial recursive function with

$$\{e\} : \{x : \mathbb{U} \models x \in n\} \to \mathbf{V}.$$

For the unique n, e such that $\alpha = sup(n, e)$ we use the shorthands the shorthands $\bar{\alpha}$ and $\tilde{\alpha}$, respectively, and write $\tilde{\alpha}(x)$ for $\{e\}(x)$.

Further, the formulae-as-types interpretation associates with each formula $\varphi(u_1, \ldots, u_r)$ of **CZF** and any a class-valued function

$$(\alpha_1, \ldots, \alpha_r) \mapsto [\![\varphi(\alpha_1, \ldots, \alpha_r)]\!]$$

(uniformly in $\alpha_1, \ldots, \alpha_r$), where $\alpha_1, \ldots, \alpha_r \in \mathbf{V}$. The interpretation is such that $\llbracket \bot \rrbracket = \emptyset$ and whenever $\mathbf{CZF} \vdash \varphi(u_1, \ldots, u_r)$, then $\mathbf{KP} \vdash \forall \alpha_1, \ldots, \alpha_r \in \mathbf{V}\ \llbracket \varphi(\alpha_1, \ldots, \alpha_r) \rrbracket \neq \emptyset$. Another fact worthwhile mentioning is that this interpretation is faithful with regard to Π^0_2 statements of arithmetic, i.e. if ψ is the set-theoretic rendering of a Π^0_2 statement of Peano arithmetic, then

$$\mathbf{KP} \vdash \llbracket \psi \rrbracket \neq \emptyset \ \leftrightarrow \ \psi.$$

To show that **MP** is validated under this interpretation, let $\varphi(u)$ be a set-theoretic formula with parameters from $\mathbf{V}$. We are to show that

$$\llbracket \forall n \in \omega\, [\varphi(n) \vee \neg\varphi(n)] \ \rightarrow\ [\neg\neg\exists n \in \omega\, \varphi(n) \rightarrow \exists n \in \omega \varphi(n)] \rrbracket \ \neq \ \emptyset. \tag{35}$$

Let $\underline{\omega}$ be the element of $\mathbf{V}$ that serves to interpret the set ω. We have $\{\underline{\widetilde{\omega}}\} : \omega \rightarrow \mathbf{V}$. For $n \in \omega$ let $\bar{n} := \{\widetilde{\omega}\}(n)$. Now suppose

$$e \in \llbracket \forall n \in \omega\, [\varphi(n) \vee \neg\varphi(n)] \rrbracket \qquad \text{and} \tag{36}$$
$$g \in \llbracket \neg\neg\exists n \in \omega\, \varphi(n) \rrbracket. \tag{37}$$

(36) yields

$$\forall n \in \omega\, (\{e\}(\bar{n}) \in \llbracket \varphi(\bar{n}) \ \vee\ \neg\varphi(\bar{n}) \rrbracket). \tag{38}$$

It is a consequence of (38) that, for some partial recursive function η and for each $n \in \omega$, either

$$\begin{array}{lll} \eta(e,n)_0 = 0 \ \wedge\ \eta(e,n)_1 \in \llbracket \varphi(\bar{n}) \rrbracket & & \text{or} \\ \eta(e,n)_0 = 1 \ \wedge\ \eta(e,n)_1 \in \llbracket \neg\varphi(\bar{n}) \rrbracket. & & \end{array}$$

As we work in the classical theory **KP**, from (37) it follows that for some $m \in \omega$, $\llbracket \varphi(\bar{m}) \rrbracket \neq \emptyset$. Consequently, there exists n such that $\eta(e,n)_0 = 0 \wedge \eta(e,n)_1 \in \llbracket \varphi(\bar{n}) \rrbracket$, and hence, with $r := \mu n.\eta(e,n)_0 = 0$, $\eta(e,r)_1 \in \llbracket \varphi(\bar{n}) \rrbracket$, so

$$\eta(e,r) \in \llbracket \exists n \in \omega\, \varphi(n) \rrbracket.$$

As a result, we have shown (35).

To show that **IP** is validated under this interpretation, assume that

$$e \in \llbracket \neg\theta \rightarrow \exists x \psi(x) \rrbracket,$$

where θ is closed. Then, if $g \in \llbracket \neg\theta \rrbracket$, we get $0 \in \llbracket \neg\theta \rrbracket$, and thus $\{e\}(0) \in \llbracket \exists x \psi(x) \rrbracket$. Therefore, with $a := (\{e\}(0))_0$ and $e' :=:= (\{e\}(0))_1$ we get $e' \in \llbracket \psi(a) \rrbracket$.

Hence, if $\llbracket \theta \rrbracket = \emptyset$, $\{e\}(0)$ is defined and

$$\{e\}(0) \in \llbracket \exists x\, (\neg\theta \rightarrow \psi(x)) \rrbracket.$$

On the other hand, should $[\![\, \theta]\!] \neq \emptyset$, then $[\![\, \neg\theta]\!] = \emptyset$, so

$$(\bar{0}, 0) \in [\![\, \exists x\, (\neg\theta \to \psi(x))]\!].$$

Thus, in every case we have shown

$$[\![\, (\neg\theta \to \exists x \psi(x)) \;\to\; \exists x\, (\neg\theta \to \psi(x))]\!] \;\; \neq \;\; \emptyset.$$

Similarly one shows that $\mathbf{IP}_\omega$ is validated under this interpretation.

The further claim that choice principles **DC**, **PAx**, and $\Pi\Sigma$-**AC** ($\Pi\Sigma W$-**AC**) may be added is a consequence of [20], Theorem 4.14 (Theorem 5.13). □

8 Absoluteness Properties

The aim of this section is to show that truth in V and realizability in $\mathrm{V}(Kl)$ mean the same for almost negative arithmetic formulae. Our first task is to single out the natural candidates for representing the natural numbers in $\mathrm{V}(Kl)$. Whenever $\exists e \in |\mathcal{A}|\; e \Vdash \varphi$, we write '$\mathrm{V}(\mathcal{A}) \models \varphi$'.

Definition: 8.1 *For $a, b \in V(Kl)$, set $\{a, b\}_{kl} := \{\langle 0, a\rangle, \langle 1, b\rangle\}$ and put*

$$\langle a, b\rangle_{Kl} \;\; := \;\; \{\langle 0, \{a, a\}_{kl}\rangle, \langle 1, \{a, b\}_{kl}\rangle\}.$$

Lemma: 8.2 *For $a, b, x \in V(Kl)$,*

$$\begin{aligned} \mathrm{V}(Kl) \;&\models\; x \in \{a, b\}_{kl} \;\leftrightarrow\; x = a \,\vee\, x = b, \\ \mathrm{V}(Kl) \;&\models\; x \in \langle a, b\rangle_{Kl} \;\leftrightarrow\; x = \{a, a\}_{kl} \,\vee\, x = \{a, b\}_{kl}. \end{aligned}$$

Proof: See [16], Lemma 3.2 and 3.4. □

Definition: 8.3 The natural internalization of ω in $\mathrm{V}(Kl)$ is defined as follows: For each $n \in \omega$, let $\overline{n} \;=\; \{\langle m, \overline{m}\rangle :\; m \in n\}$ and set

$$\overline{\omega} \;\; = \;\; \{\langle n, \overline{n}\rangle :\; n \in \omega\}.$$

Proposition: 8.4 (**CZF**) *Equality and membership are realizably absolute for $\overline{\omega}$. This means that for all $n, m \in \omega$,*

$$\begin{aligned} m = n \quad &\textit{iff} \quad V(Kl) \models \overline{m} = \overline{n} \qquad \textit{and} \\ m \in n \quad &\textit{iff} \quad V(Kl) \models \overline{m} \in \overline{n}. \end{aligned}$$

Proof: [16], Ch.3, Theorem 3.11. $\square$

Elementary recursion theory can be formalized in Heyting arithmetic (cf. [22], Vol. Ch.3, section 6) and a fortiori it can be formalized in **CZF**. In particular one can talk about primitive recursive relations in **CZF**. Each primitive recursive n-ary relation R on ω is canonically represented by a formula φ_R in the language of **CZF**. In the following we shall write $\mathrm{V}(Kl) \models R(\overline{n_1}, \ldots, \overline{n_r})$ rather than the more accurate $\mathrm{V}(Kl) \models \varphi_R(\overline{n_1}, \ldots, \overline{n_r})$.

Proposition: 8.5 (**CZF**). *When R is a primitive recursive r-ary relation R on ω and $n_1, \ldots, n_r \in \omega$, then*

$$R(n_1, \ldots, n_r) \quad \textit{iff} \quad V(Kl) \models R(\overline{n_1}, \ldots, \overline{n_r}).$$

Proof: See [16], Ch.4, Theorem 2.6. $\square$

Definition: 8.6 A formula θ of the language of **CZF** which contains solely parameters from ω is said to be *almost negative arithmetic* if it is built from primitive recursive formulas φ_R, using the connectives $\wedge, \rightarrow, \neg$, bounded universal quantifiers $\forall n \in \omega$, and bounded existential quantifiers $\exists m \in \omega$ which appear only as prefixed to primitive recursive subformulae of θ.

Theorem: 8.7 (**CZF**). *If $n_1, \ldots, n_r \in \omega$ and $\theta(n_1, \ldots, n_r)$ is an almost negative arithmetic formula, then there is an r-place primitive recursive function f_θ such*

$$\begin{aligned} \theta(n_1, \ldots, n_r) \quad & \textit{iff} \quad f_\theta(n_1, \ldots, n_r) \Vdash \theta(\overline{n_1}, \ldots, \overline{n_r}) \\ & \textit{iff} \quad V(Kl) \models \theta(\overline{n_1}, \ldots, \overline{n_r}). \end{aligned}$$

Proof: This is proved by induction on the build-up of θ. For the primitive recursive subformulas this follows from the proof of [16], Ch.4, Theorem 2.6. For the inductive steps one proceeds exactly as in the proof of [22], Sect.4, Proposition 4.5. $\square$

9 Some classical and non-classical principles that hold in V(Kl)

The next definitions lists several interesting principles that are validated in $\mathrm{V}(Kl)$.

Definition: 9.1 1. *Church's Thesis*, **CT**, is formalized as

$$\forall n \in \omega\, \exists m \in \omega\, \varphi(n, m) \;\rightarrow\; \exists e \in \omega\, \forall n \in \omega\, \exists m, p \in \omega\, [T(e, n, p) \wedge U(p, m) \wedge \varphi(n, m)]$$

for every formula $\varphi(u, v)$, where T and U are the set-theoretic predicates which numeralwise represent, respectively, Kleene's T and result-extraction predicate U.

2. *Extended Church's Thesis*, **ECT**, asserts that

$$\forall n \in \omega\, [\psi(n) \to \exists m \in \omega\, \varphi(n,m)] \qquad \text{implies}$$
$$\exists e \in \omega\, \forall n \in \omega\, [\psi(n) \to \exists m, p \in \omega\, [T(e,n,p) \wedge U(p,m) \wedge \varphi(n,m)]]$$

   whenever $\psi(n)$ is an almost negative arithmetic formula and $\varphi(u,v)$ is any formula. Recall that formula θ of the language of **CZF** with quantifiers ranging over ω is said to be *almost negative arithmetic* if $\vee$ does not appear in it and instances of $\exists m \in \omega$ appear only as prefixed to primitive recursive subformulae of θ.

   Note that **ECT** implies **CT**, taking $\psi(n)$ to be $n = n$.

3. **UP**, the *Uniformity Principle*, is expressed by the scheme:

$$\forall x\, \exists n \in \omega\, \varphi \;\to\; \exists n \in \omega\, \forall x \varphi.$$

4. *Unzerlegbarkeit*, **UZ**, is the scheme

$$\forall x(\phi \vee \neg\phi) \;\to\; \forall x\varphi \vee \forall x\neg\varphi$$

   for all formulas φ.

A set is said to be *subcountable* if it is the surjective image of a subset of ω.

It is known that all the above principles hold in $\mathrm{V}(Kl)$ if one assumes the axioms of **IZF** in the background universe V (see [16]). Owing to results of sections 5 and 6, we know that Kleene realizability is self-validating for **CZF** and **CZF** + **REA**. By inspection of the proofs of [16] one arrives at the following theorem:

Theorem: 9.2 (**CZF**) $\mathrm{V}(Kl) \models \mathbf{AC}^{\omega,\omega} \wedge \mathbf{CT} \wedge \mathbf{ECT} \wedge \mathbf{UP} \wedge \mathbf{UZ}$.

Proof: (1): The proof of $\mathrm{V}(Kl) \models \mathbf{AC}^{\omega,\omega}$ in Chap.3, Theorem 5.1, [16] uses intuitionistic logic and besides that just means available in **CZF**.

(2): One readily checks that the proof Theorem 3.1, [16] of $\mathrm{V}(Kl) \models \mathbf{CT}$ just utilizes $\mathrm{V}(Kl) \models \mathbf{AC}^{\omega,\omega}$ and axioms from **CZF**.

(3): Inspection of Theorem 8.2, [16] shows that the proof of $\mathrm{V}(Kl) \models \mathbf{UP}$ carries over to **CZF**.

(4): From (3) one immediately deduces that

$$\mathrm{V}(Kl) \models y \textit{ is subcountable}$$

implies

$$\mathrm{V}(Kl) \models \forall x\, \exists z \in y\, \varphi \;\to\; \exists z \in y\, \forall x\, \varphi.$$

Now, since any two element set is subcountable, it follows from the above that $\mathrm{V}(Kl) \models$ **UZ**.

Ad **ECT**: Assume $d \Vdash \forall n \in \omega\, [\psi(n) \to \exists m \in \omega\, \varphi(n,m)]$, where $\psi(n)$ is an almost negative arithmetic formula. Invoking Theorem 8.7, there is a number $t_\psi(n) \in \omega$ depending primitive recursively on n (and possibly further parameters of ψ), such that

$$\begin{aligned} \mathrm{V}(Kl) \models \psi(n) \quad & \text{iff} \quad t_\psi \Vdash \psi(n) \\ & \text{iff} \quad \psi(n), \end{aligned}$$

so that with $f(n) = (\lambda u.du)t_\psi(n)$ we arrive at

$$\forall n \in \omega\, [\psi(n) \to f(n) \Vdash \exists m \in \omega\, \varphi(\overline{n}, m)].$$

From the latter it follows with $h(n) = (f(n))_0$ and $l(n) = f(n)_1$ that

$$\forall n \in \omega\, [\psi(n) \to l(n) \Vdash \varphi(\overline{n}, \overline{h(n)})].$$

Taking e to be an index for h, it is obvious that from n we can effectively construct an realizer $\rho(n)$ such that

$$\forall n \in \omega\, [\psi(n) \to \rho(n) \Vdash \exists m, p \in \omega\, [T(\overline{e}, \overline{n}, p) \wedge U(p, m) \wedge \varphi(\overline{n}, m)]].$$

Hence, owing to Theorem 8.7, we can calculate effectively from d a realizer d' such that

$$d' \Vdash \exists e \in \omega\, \forall n \in \omega\, [\psi(n) \to \rho(n) \Vdash \exists m, p \in \omega\, [T(e, n, p) \wedge U(p, m) \wedge \varphi(n, m)]].$$

□

The principles **MP** and **IP** are known to propagate from V to V(Kl). We therefore obtain the following results:

Theorem: 9.3 *1.* (**CZF** + **MP**) V$(Kl) \models$ **MP**.

2. (**CZF** + **IP**) V$(Kl) \models$ **IP**.

Proof: See [16], chap.3, Theorem 11.3 and Theorem 11.5. □

10 Axioms of choice and V(Kl)

While $\mathbf{AC}^{\omega,\omega}$ holds in V(Kl) for free, i.e. without assuming any choice in the background universe, validity of the following choice principles in V(Kl) seems to require their respective validity in V.

Theorem: 10.1 *(i)* (**CZF** + **DC**) V$(Kl) \models$ **DC**.

(ii) (**CZF** + **RDC**) V$(Kl) \models$ **RDC**.

(iii) (**CZF** + **PAx**) V$(Kl) \models$ **PAx**.

Proof: We shall use the abbreviations (x,y), $(x)_0$, and $(x)_1$ for $\mathbf{p}^{Kl}xy$, $\mathbf{p_0}^{Kl}x$, and $\mathbf{p_1}^{Kl}x$, respectively.

(i): This proof is similar to the proof of [16], chap.3, Theorem 6.1. However, because of the special way we defined realizability for bounded quantifiers there are some differences. In particular, the set $a^{\Vdash}$ used in the proof of that Theorem will not be needed here.

Let $a, u \in \mathrm{V}(Kl)$. Suppose

$$\begin{aligned} &e \Vdash \forall x \in a\, \exists y \in a\, \varphi(x,y) \qquad \text{and} \\ &e^* \Vdash u \in a. \end{aligned}$$

Then there exists $\langle k_0, u_0\rangle \in a$ such that $k_0 = (e^*)_0$ and $(e^*)_1 \Vdash u = u_0$ and

$$\forall \langle n, x\rangle \in a\, \exists y\, [\langle (\{e\}(n))_0, y\rangle \in a \ \wedge\ ((\{e\}(n))_1 \Vdash \varphi(x,y)].$$

Thus, for all $n \in \omega$ and all $b \in \mathrm{V}(Kl)$, if $\langle n, b\rangle \in a$, then $\{e\}(n)\downarrow$ and there is a $c \in \mathrm{V}(Kl)$ such that

$$\langle (\{e\}(n))_0, c\rangle \in a \quad \text{and} \quad (\{e\}(n))_1 \Vdash \varphi(b,c).$$

Let $\varphi^{\Vdash}$ be such that, externally,

$$\varphi^{\Vdash}(\langle n, b\rangle, \langle m, c\rangle) \quad \text{iff} \quad m = (\{e\}(n))_0 \ \wedge\ (\{e\}(n))_1 \Vdash \varphi(b,c).$$

By **DC** in the ground model, there is a function $F : \omega \to a$ such that

$$F(0) = \langle k_0, u_0\rangle \quad \text{and} \quad \forall n \in \omega\ \varphi^{\Vdash}(F(n), F(n+1)).$$

Next, we internalize F and prove that it supplies the function required for the truth of **DC** in $\mathrm{V}(Kl)$. If x is an ordered pair $\langle u, v\rangle$, let $(x)^s_0 = u$ and $(x)^s_1 = v$. The appropriate internalization of F is $\overline{F}$:

$$\overline{F} \ = \ \{\langle (n, (F(n))^s_0), \langle \overline{n}, (F(n))^s_1\rangle_{Kl}\rangle \ :\ n \in \omega\}.$$

Obviously, $\overline{F}$ belongs to $\mathrm{V}(Kl)$. It remains to check that $\overline{F}$ is internally a function from ω into a, and that $\overline{F}$ makes the consequent of the pertinent instance of **DC** realizable as well. This part of the proof is almost the same as that of [16], chap.3, Theorem 6.1, However, for the readers convenience and for later reference, we shall provide the details all the same.

First, because of the properties of internal pairing in $\mathrm{V}(Kl)$ (cf. [16], chap.3, Lemma 3.4), it will be shown that $\mathrm{V}(Kl)$ believes that $\overline{F}$ is a binary relation with domain $\overline{\omega}$ and range a subset of a and that this holds with a witness obtainable independently of e and e^*. To see that $\overline{F}$ is realizably functional, assume that

$$h \Vdash \langle x, y\rangle_{Kl} \in \overline{F} \quad \text{and} \quad j \Vdash \langle x, z\rangle_{Kl} \in \overline{F}.$$

Then,

$$h_1 \Vdash \langle x, y\rangle_{Kl} = \langle \overline{h_{00}}, (F(h_{00}))^s_1\rangle_{Kl} \quad \text{and} \quad j_1 \Vdash \langle x, z\rangle_{Kl} = \langle \overline{j_{00}}, (F(j_{00}))^s_1\rangle_{Kl},$$

where $h_1 = (h)_1$, $h_{00} = ((h)_0)_0$, $j_1 = (j)_1$, and $j_{00} = ((j)_0)_0$. This holds strictly in virtue of the definition of $\overline{F}$ and the conditions on statements of membership. From the absoluteness of $\in$ and $=$ on ω (see Proposition 8.4), we know that $h_{00} = j_{00}$ and thus $(F(h_{00}))^s_1 = (F(j_{00}))^s_1$. Therefore, there is a partial recursive function ϑ such that $\vartheta(h, j) \Vdash y = z$. ϑ confirms that $\overline{F}$ is realizably functional. Next, to see that $\mathrm{V}(Kl) \models \overline{F} \subseteq \overline{\omega} \times a$, let

$$h \Vdash \langle x, y\rangle_{Kl} \in \overline{F}.$$

As above,

$$h_1 \Vdash \langle x, y\rangle_{Kl} = \langle \overline{h_{00}}, (F(h_{00}))^s_1\rangle_{Kl}.$$

Moreover, $(h_{00}, \mathbf{i}_r) \Vdash \overline{h_{00}} \in \overline{\omega}$, and, by definition of $\overline{F}$, $(h_{01}, \mathbf{i}_r) \Vdash (F(h_{00}))^s_1 \in a$. As a result, we can effectively compute $h^*, h^\#$ from h such that $h^* \Vdash x \in \overline{\omega}$ and $h^\# \Vdash y \in a$.

Finally, we have to check on the realizability of $\overline{F}(0) = u$ and $\forall n \in \overline{\omega}\, \varphi(\overline{F}(n), \overline{F}(n{+}1))$. Obviously, $((0, (e^*)_0), \mathbf{i}_r) \Vdash \langle 0, u_0\rangle_{Kl} \in \overline{F}$; thus from e^* we can effectively compute a number e^{**} such that $e^{**} \Vdash \overline{F}(0) = u$. Since, for all $n \in \omega$, $\varphi^{\Vdash}(F(n), F(n+1))$, we have for all $n \in \omega$,

$$\begin{array}{rcl}(\{e\}((F(n))^s_0))_0 & = & (F(n+1))^s_0 \quad \text{and} \\ (\{e\}((F(n))^s_0))_1 & \Vdash & \varphi((F(n))^s_1, (F(n+1))^s_1).\end{array}$$

Define the recursive function ρ so that

$$\rho(0) = (e^*)_0 \quad \text{and } \rho(n+1) = (\{e\}(\rho(n)))_0.$$

The S-m-n theorem shows that an index for ρ is calculable from e and e^*. Then, one can use induction over ω to check that, for all $n \in \omega$,

$$\begin{array}{rcl}((n, \rho(n)), \mathbf{i}_r) & \Vdash & \langle \overline{n}, (F(n))^s_1\rangle_{Kl} \in \overline{F} \quad \text{and} \\ \{e\}(\rho(n)) & \Vdash & \varphi((F(n))^s_1, (F(n+1))^s_1).\end{array}$$

This completes the proof of (i).

(ii): **RDC** implies **DC** (see [21], Lemma 3.4) and, on the basis of **CZF** + **DC**, **RDC** follows from the following scheme:

$$\begin{array}{l}\forall x\, (\varphi(x) \to \exists y\, [\varphi(y) \wedge \psi(x, y)]) \wedge \varphi(b) \quad \to \\ \qquad\qquad \exists z\, (b \in z \wedge \forall x \in z\, \exists y \in z\, [\varphi(y) \wedge \psi(x, y)]).\end{array} \tag{39}$$

Thus, in view of part (i) of this theorem it suffices to show that, working in **CZF** + **RDC**, $\mathrm{V}(Kl)$ validates (39). So let $a, b \in \mathrm{V}(Kl)$ and suppose

$$\begin{array}{rcl}e & \Vdash & \forall x\, (\varphi(x) \to \exists y\, [\varphi(y) \wedge \psi(x, y)]) \qquad \text{and} \\ g & \Vdash & \varphi(b).\end{array}$$

Therefore, for all $f \in \omega$ and $x \in \mathrm{V}(Kl)$ we have

$$f \Vdash \varphi(x) \;\rightarrow\; \exists y \in \mathrm{V}(Kl)\,[(\{e\}(f))_0 \Vdash \varphi(y) \;\wedge\; (\{e\}(f))_1 \Vdash \psi(x,y)].$$

By applying **RDC** to the above, one can extract functions $\imath : \omega \rightarrow \omega$, $\jmath : \omega \rightarrow \omega$, and $\ell : \omega \rightarrow \mathrm{V}(Kl)$ such that $\imath(0) = g$, $\ell(0) = b$, and for all $n \in \omega$:

$$\begin{array}{ccc} \imath(n) \Vdash \varphi(\ell(n)) & \text{and} & \jmath(n) \Vdash \psi(\ell(n), \ell(n+1)), \\ \imath(n+1) = (\{e\}(\imath(n)))_0 & \text{and} & \jmath(n) = (\{e\}(\imath(n)))_1. \end{array}$$

By the last line, $\imath$ and $\jmath$ are recursive functions whose indices can be effectively computed from e and g. Thus, defining $\hbar : \omega \rightarrow \omega$ by

$$\hbar(n) \;=\; (n, (\imath(n), \jmath(n))),$$

$\hbar$ is a recursive function the index of which is calculable from e and g. Now set

$$B \;=\; \{\langle \hbar(n), \ell(n) \rangle \;:\; n \in \omega\}.$$

Obviously, B belongs to $\mathrm{V}(Kl)$. We have

$$(\hbar(0), \mathbf{i}_r) \Vdash b \in B. \tag{40}$$

Moreover, if $\langle k, u \rangle \in B$, then $u = \ell((k)_0)$ and $\langle \hbar((k)_0 + 1), \ell((k)_0 + 1) \rangle \in B$ and

$$((k)_1)_0 \Vdash \varphi(u) \;\wedge\; ((k)_1)_1 \vdash \psi(u, \ell((k)_0 + 1)).$$

Thus,

$$\forall \langle k, u \rangle \in B\, \exists v\, \langle \hbar((k)_0 + 1), v \rangle \in B\, [((k)_1)_0 \Vdash \varphi(u) \;\wedge\; ((k)_1)_1 \vdash \psi(u, v)]. \tag{41}$$

In view of (40) and (41) it is obvious that there is an index $e^{\#}$ calculable from e and g such that

$$e^{\#} \Vdash b \in B \;\wedge\; \forall x \in B\, \exists y \in B\, [\varphi(x) \;\wedge\; \psi(x,y)],$$

whence

$$e^{\#} \Vdash \exists z\, (b \in z \;\wedge\; \forall x \in z\, \exists y \in z\, [\varphi(x) \;\wedge\; \psi(x,y)]).$$

This finishes the proof of (ii)

(iii): The proof of $\mathrm{V}(Kl) \models$ **PAx** given in [16], chap.3, Theorem 7.6 assumes the full axiom of choice to hold in V and thereby appears to be requiring nothing less than the means of **ZFC**. In consequence, we have to find an entirely different proof.

Now let $a \in \mathrm{V}(Kl)$. We have to find a set $B^* \in \mathrm{V}(Kl)$ such that $\mathrm{V}(Kl)$ thinks that B^* is a base that surjects onto a. Because **PAx** holds in the background universe, we can select a base B and a surjection $\jmath : B \rightarrow a$. a being a set of pairs, define $\jmath_0 : B \rightarrow \omega$ and $\jmath_1 : B \rightarrow \mathrm{V}(Kl)$ by

$$\begin{array}{rcl} \jmath_0(u) & = & \mathit{First}(\jmath(u)), \\ \jmath_1(u) & = & \mathit{Second}(\jmath(u)), \end{array}$$

where for an ordered pair $z = \langle x, y\rangle$, $First(z) = x$ and $Second(z) = y$.

By transfinite recursion define

$$x^{st} \quad = \quad \{\langle 0, y^{st}\rangle \,:\, y \in x\}$$

for each set x. By $\in$-induction, $x^{st} \in \mathrm{V}(Kl)$, and by a simultaneous $\in$-induction (see [16],chap.3, 10.2),

$$\begin{aligned} x = y \quad &\text{iff} \quad \mathrm{V}(Kl) \models x^{st} = y^{st} \\ x \in y \quad &\text{iff} \quad \mathrm{V}(Kl) \models x^{st} \in y^{st}; \end{aligned} \tag{42}$$

thus $(x \mapsto x^{st})$ injects V into $\mathrm{V}(Kl)$. Now, define

$$B^* \quad = \quad \{\langle \jmath_0(u), \langle \overline{\jmath_0(u)}, u^{st}\rangle_{Kl}\rangle \;:\; u \in B\}.$$

First, note that B^* is in one-one correspondence with B via $u \mapsto \langle \jmath_0(u), \langle \overline{\jmath_0(u)}, u^{st}\rangle_{Kl}\rangle$ (owing to (42)), and hence (externally in the background universe) B^* is a base as well. Let $\ell : B \to \mathrm{V}(Kl)$ be defined by $\ell(u) = \langle \overline{\jmath_0(u)}, u^{st}\rangle_{Kl}$ and put

$$\jmath^* \quad = \quad \{\langle \jmath_0(u), \langle \ell(u), \jmath_1(u)\rangle_{Kl}\rangle \;:\; u \in B\}.$$

Clearly, $\jmath^* \in \mathrm{V}(Kl)$. First, we aim at showing that

$$\mathrm{V}(Kl) \models \jmath^* \textit{ is a surjection from } B^* \textit{ onto } a. \tag{43}$$

To verify $\mathrm{V}(Kl) \models \jmath^* \subseteq B^* \times a$, suppose $e \Vdash \langle b, c\rangle_{Kl} \in \jmath^*$. Then there is a $u \in B$ such that

$$(e)_0 = \jmath_0(u) \quad \text{ and } \quad (e)_1 \Vdash \langle b, c\rangle_{Kl} = \langle \ell(u), \jmath_1(u)\rangle_{Kl}.$$

Hence, because of

$$(\jmath_0(u), \mathbf{i}_r) \Vdash \ell(u) \in B^* \quad \text{ and } \quad (\jmath_0(u), \mathbf{i}_r) \Vdash \jmath_1(u) \in a,$$

one can effectively calculate an index e' from e such that $e' \Vdash b \in B^* \wedge c \in a$. This shows

$$\mathrm{V}(Kl) \models \jmath^* \subseteq B^* \times a. \tag{44}$$

To see that $\jmath^*$ is realizably total on B^*, let $e \Vdash \langle c, d\rangle_{Kl} \in B^*$. Then $(e)_0 = \jmath_0(u)$ and $(e)_1 \Vdash \langle c, d\rangle_{Kl} = \ell(u)$ for some $u \in B$. Since $(\jmath_0(u), \mathbf{i}_r) \Vdash \jmath_1(u) \in a$ and $(\jmath_0(u), \mathbf{i}_r) \Vdash \langle \ell(u), \jmath_1(u)\rangle_{Kl} \in \jmath^*$, an index $\tilde{e}$ can be computed from e such that

$$\tilde{e} \Vdash \langle c, d\rangle_{Kl} \textit{ is in the domain of } \jmath^*$$

so that with (44) we can conclude that $\mathrm{V}(Kl) \models B^*$ *is the domain of* $\jmath^*$.

To show realizable functionality of $\jmath^*$, suppose $f \Vdash \langle b, c\rangle_{Kl} \in \jmath^*$ and $h \Vdash \langle b, d\rangle_{Kl} \in \jmath^*$. Then there exist $u, v \in B$ such that $(f)_0 = \jmath_0(u)$, $(h)_0 = \jmath_0(v)$, $(f)_1 \Vdash \langle b, c\rangle_{Kl} = \langle \ell(u), \jmath_1(u)\rangle_{Kl}$, and $(h)_1 \Vdash \langle b, d\rangle_{Kl} = \langle \ell(v), \jmath_1(v)\rangle_{Kl}$. From the latter one gets $\mathrm{V}(Kl) \models \ell(u) = \ell(v)$. Given the definition of ℓ and the properties of internal pairing one concludes with the aid of (42) that actually $u = v$. In consequence, an index e' such that $e' \Vdash c = d$ is calculable from f and h.

Next, to show that $\jmath^*$ realizably maps onto a, assume $e \Vdash x \in a$. Then $\langle (e)_0, c\rangle \in a$ and $(e)_1 \Vdash x = c$ for some $c \in \mathrm{V}(Kl)$. As $\jmath$ maps B onto a there exists $u \in B$ such that $\jmath_0(u) = (e)_0$ and $\jmath_1(u) = c$. Moreover, because of $(\jmath_0(u), \mathbf{i}_r) \Vdash \ell(u) \in B^*$ and $(\jmath_0(u), \mathbf{i}_r) \Vdash \langle \ell(u), \jmath_1(u)\rangle_{Kl} \in \jmath^*$, one can effectively construct an index $\tilde{e}$ from e such that $\tilde{e} \Vdash x$ *is in the range of* $\jmath^*$, thereby completing the proof of (43).

It remains to ensure that $\mathrm{V}(Kl)$ thinks that B^* is a base. Towards this goal, assume

$$e \Vdash \forall x \in B^* \, \exists y \, \varphi(x, y) \tag{45}$$

for some formula φ. We are to determine an index e' calculably from e that satisfies

$$e' \Vdash \exists G \, [\mathbf{Fun}(G) \;\wedge\; \mathbf{dom}(G) = B^* \;\wedge\; \forall x \in B^* \, \varphi(x, G(x))]. \tag{46}$$

From (45) it follows that

$$\forall \, \langle n, c\rangle \in B^* \, \exists y \in \mathrm{V}(Kl) \; \{e\}(n) \Vdash \varphi(c, y).$$

Since B^* is a base in the background universe, there exists a function $F : B^* \to \mathrm{V}(Kl)$ such that with $F(n, c) := F(\langle n, c\rangle)$,

$$\forall \, \langle n, c\rangle \in B^* \; \{e\}(n) \Vdash \varphi(c, F(n, c)). \tag{47}$$

Next, we want to internalize F. The appropriate internalization of F is $\tilde{F}$:

$$\tilde{F} \;\; = \;\; \{\langle (\{e\}(n), n), \langle c, F(n, c)\rangle_{Kl}\rangle \; : \; \langle n, c\rangle \in B^*\}.$$

Obviously, $\tilde{F}$ belongs to $\mathrm{V}(Kl)$.

First we show that there exists an index $\hat{e}$ calculable from e such that

$$\hat{e} \;\; \Vdash \;\; \mathbf{dom}(\tilde{F}) = B^*. \tag{48}$$

To this end assume that $h \Vdash x \in B^*$. Then $\langle (h)_0, c\rangle \in B^*$ and $(h)_1 \Vdash x = c$ for some $c \in \mathrm{V}(Kl)$. From $\langle (h)_0, c\rangle \in B^*$ it follows that

$$((\{e\}(((h)_0), (h)_0), \mathbf{i}_r) \Vdash \langle c, F((h)_0, c)\rangle_{Kl} \in \tilde{F}, \tag{49}$$

and hence we can compute an index h^* from h such that $h^* \Vdash x \in \mathbf{dom}(\tilde{F})$. Conversely, suppose $d \Vdash \langle x, y\rangle_{Kl} \in \tilde{F}$. Then there exists $\langle n, c\rangle \in B^*$ such that $\langle (d)_0, \langle c, F(n, c)\rangle_{Kl}\rangle \in \tilde{F}$ with $n = ((d)_0)_1$ and $(d)_1 \Vdash \langle x, y\rangle_{Kl} = \langle c, F(n, c)\rangle_{Kl}$. Thus $(((d)_0)_1, \mathbf{i}_r) \Vdash c \in B^*$. In consequence, there is an index d^* calculable from d such that $d^* \Vdash x \in B^*$. Therefore we have all the ingredients to compose $\hat{e}$ as claimed in (48).

To show realizable functionality of $\tilde{F}$, suppose $f \Vdash \langle b, c\rangle_{Kl} \in \tilde{F}$ and $h \Vdash \langle b, d\rangle_{Kl} \in \tilde{F}$. Then there exist $\langle n, x\rangle, \langle m, y\rangle \in B^*$ such that $((f)_0)_1 = n$, $((h)_0)_1 = m$, and

$$(f)_1 \Vdash \langle b, c\rangle_{Kl} = \langle x, F(n, x)\rangle_{Kl} \quad \wedge \quad (h)_1 \Vdash \langle b, d\rangle_{Kl} = \langle y, F(m, y)\rangle_{Kl}. \tag{50}$$

From (50) one gets $\mathrm{V}(Kl) \models x = y$. Moreover, as $\langle n, x\rangle, \langle m, y\rangle \in B^*$ there are $u, v \in B$ such that

$$\begin{array}{lclcl} x = \langle\overline{\jmath_0(u)}, u^{st}\rangle_{Kl} & \text{and} & n = \jmath_0(u), & \text{and} \\ y = \langle\overline{\jmath_0(v)}, v^{st}\rangle_{Kl} & \text{and} & m = \jmath_0(v). \end{array}$$

Since $\mathrm{V}(Kl) \models x = y$, the foregoing yields $\mathrm{V}(Kl) \models \overline{n} = \overline{m} \ \wedge \ u^{st} = v^{st}$, and so, by Proposition 8.4 and (42), we can conclude that $n = m$ and $u = v$, and hence $x = y$ and $F(n, x) = F(m, y)$. Thus, in view of (50), there is a partial recursive function ν such that $\nu(f, h) \Vdash c = d$, verifying functionality of $\tilde{F}$, i.e., $\mathrm{V}(Kl) \models \tilde{F}$ *is a function.*

Combining the latter result with (48) and (47) allows one to construct the desired e' from e such that (46) holds. $\square$

11 Continuity Principles

Fundamental to Brouwer's development of intuitionistic mathematics are strong continuity principles incompatible with classical mathematics.

Definition: 11.1 Some continuity principles which pertain to Brouwer's mathematics are:[1]

1. $\mathbf{Cont}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$: *Every function from* $\mathbb{N}^{\mathbb{N}}$ *to* $\mathbb{N}$ *is continuous.*
2. $\forall \mathbb{X} \forall \mathbb{Y}\, \mathbf{Cont}(\mathbb{X}, \mathbb{Y})$: *For every complete separable metric space* $\mathbb{X}$ *and separable metric space* $\mathbb{Y}$*,* $\mathbf{Cont}(\mathbb{X}, \mathbb{Y})$*,* i.e., every function from $\mathbb{X}$ to $\mathbb{Y}$ is continuous.

Recall that $\mathbf{MP_{PR}}$ is Markov's principle for primitive recursive predicates.

Theorem: 11.2 $(\mathbf{CZF} + \mathbf{MP_{PR}})$. $\mathrm{V}(Kl) \models \mathbf{Cont}(\mathbb{N}^{\mathbb{N}}, \mathbb{N}) \ \wedge \ \forall \mathbb{X} \forall \mathbb{Y}\, \mathbf{Cont}(\mathbb{X}, \mathbb{Y})$.
Moreover, V(Kl) *validates that every separable metric space is subcountable.*

Proof: For the proof that $\mathbf{MP}$ is realized in $\mathrm{V}(Kl)$ it suffices to have $\mathbf{MP_{PR}}$ in the background universe. As a result of this and Theorem 9.2, we have $\mathrm{V}(Kl) \models \mathbf{MP} \wedge \mathbf{ECT}$.

By [7] IV.3.1, $\mathbf{MP_{PR}}$ proves $\mathbf{KLS}$, where $\mathbf{KLS}$ stands for the Kreisel-Lacombe-Shoenfield's theorem asserting that every effective operation from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ is continuous. By [7] XVI.2.1.1, $\mathbf{ECT}$ implies $\mathbf{KLS} \to \mathbf{Cont}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$. In consequence, $\mathrm{V}(Kl) \models \mathbf{Cont}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$.

Next, let $\mathbb{X}$ be a complete separable metric space and let $\mathbb{Y}$ be a separable metric space. By [7] I.20.1, every complete separable metric space $\mathbb{X} = (X, \sigma)$ is (isometric to) a space of the form $X = \{x \in \mathbb{N}^{\mathbb{N}} : \ \forall n, m[\rho(x_n, x_m) < 1/n + 1/m]\}$, where ρ is a metric on $\mathbb{N}$, and the metric σ on X is given by $\sigma(x, y) = \lim_{n\to\infty} \rho(x_n, y_n)$. Employing Church's thesis, ρ is recursive, and X is identified with a certain set of total indices. Note also that thereby

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[1] For exact formalizations of the notions of complete metric space, separable metric space, and continuity in constructive set theory see [7], chap. I, section 20.

the formula $x \in X$ is rendered almost negative. If $\mathbb{Y}$ is a separable metric space, then $\mathbb{Y}$ has a completion which is a complete separable metric space, and so can be identified with a subset of $\mathbb{N}^{\mathbb{N}}$ as above. Under Church's thesis, $\mathbb{N}^{\mathbb{N}}$ can be identified with a subset of $\mathbb{N}$. So every separable metric space is isometric to a subset of $\mathbb{N}$ with a recursive metric. As a result, we get $\mathbf{KLS}(\mathbb{X}, \mathbb{Y})$, i.e. every effective operation from $\mathbb{X}$ to $\mathbb{Y}$ is continuous (cf. [7] IV.3). Under **ECT**, $\mathbf{KLS}(\mathbb{X}, \mathbb{Y})$ implies $\mathbf{Cont}(\mathbb{X}, \mathbb{Y})$ by [7] XVI.2.1.1. So the upshot of the above is that the model $\mathrm{V}(Kl)$ validates $\forall\mathbb{X}\forall\mathbb{Y}\,\mathbf{Cont}(\mathbb{X}, \mathbb{Y})$ as well. In the course of the proof it was also shown that $\mathrm{V}(Kl)$ thinks that every separable metric space is subcountable. □

References

[1] P. Aczel: *The type theoretic interpretation of constructive set theory.* In: MacIntyre, A. and Pacholski, L. and Paris, J, editor, *Logic Colloquium '77* (North Holland, Amsterdam 1978) 55–66.

[2] P. Aczel: *The type theoretic interpretation of constructive set theory: Choice principles.* In: A.S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium* (North Holland, Amsterdam 1982) 1–40.

[3] P. Aczel: *The type theoretic interpretation of constructive set theory: Inductive definitions.* In: R.B. et al. Marcus, editor, *Logic, Methodology and Philosophy of Science VII* (North Holland, Amsterdam 1986) 17–49.

[4] P. Aczel, M. Rathjen: *Notes on constructive set theory*, Technical Report 40, Institut Mittag-Leffler (The Royal Swedish Academy of Sciences, 2001). *http://www.ml.kva.se/preprints/archive2000-2001.php*

[5] J. Barwise: *Admissible Sets and Structures* (Springer-Verlag, Berlin, Heidelberg, New York, 1975).

[6] M. Beeson: *Continuity in intuitionistic set theories*, in: M Boffa, D. van Dalen, K. McAloon (eds.): *Logic Colloquium '78* (North-Holland, Amsterdam, 1979).

[7] M. Beeson: *Foundations of Constructive Mathematics.* (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985).

[8] E. Bishop and D. Bridges: *Constructive Analysis.* (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985).

[9] S. Feferman: *A language and axioms for explicit mathematics*, in: J.N. Crossley (ed.): *Algebra and Logic*, Lecture Notes in Math. 450 (Springer, Berlin 1975) 87–139.

[10] S. Feferman: *Constructive theories of functions and classes* in: Boffa, M., van Dalen, D., McAloon, K. (eds.), *Logic Colloquium '78* (North-Holland, Amsterdam 1979) 159–224.

[11] H. Friedman: *Some applications of Kleene's method for intuitionistic systems.* In: A. Mathias and H. Rogers (eds.): *Cambridge Summer School in Mathematical Logic*, volume 337 of *Lectures Notes in Mathematics* (Springer, Berlin, 1973) 113–170.

[12] S.C. Kleene: *On the interpretation of intuitionistic number theory.* Journal of Symbolic Logic 10 (1945) 109–124.

[13] G. Kreisel and A.S. Troelstra: *Formal systems for some branches of intuitionistic analysis.* Annals of Mathematical Logic 1 (1970) 229–387.

[14] P. Martin-Löf: *An intuitionistic theory of types: predicative part*, in: H.E. Rose and J. Sheperdson (eds.): *Logic Colloquium '73* (North-Holland, Amsterdam, 1975) 73–118.

[15] P. Martin-Löf: *Intuitionistic Type Theory*, (Bibliopolis, Naples, 1984).

[16] D.C. McCarty: *Realizability and recursive mathematics*, PhD thesis, Oxford University (1984), 281 pages.

[17] J. Myhill: *Some properties of Intuitionistic Zermelo-Fraenkel set theory.* In: A. Mathias and H. Rogers (eds.): *Cambridge Summer School in Mathematical Logic*, volume 337 of *Lectures Notes in Mathematics* (Springer, Berlin, 1973) 206–231.

[18] J. Myhill: *Constructive set theory.* Journal of Symbolic Logic, 40:347–382, 1975.

[19] M. Rathjen: *Fragments of Kripke-Platek set theory with infinity,* in: P. Aczel, H. Simmons, S. Wainer (eds.): *Proof Theory* (Cambridge University Press, Cambridge, 1992) 251–273.

[20] M. Rathjen: *The strength of some Martin-Löf type theories.* Archive for Mathematical Logic, 33:347–385, 1994.

[21] M. Rathjen: *The anti-foundation axiom in constructive set theories.* In: G. Mints, R. Muskens (eds.): *Games, Logic, and Constructive Sets.* (CSLI Publications, Stanford, 2003) 87–108.

[22] A.S. Troelstra and D. van Dalen: *Constructivism in Mathematics, Volumes I, II.* (North Holland, Amsterdam, 1988).