

# A TRANSFERENCE PRINCIPLE FOR GENERAL GROUPS AND FUNCTIONAL CALCULUS ON UMD SPACES

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ABSTRACT. Let  $-iA$  be the generator of a  $C_0$ -group  $(U(s))_{s \in \mathbb{R}}$  on a Banach space  $X$ , and  $\omega > \theta(U)$ . We prove a transference principle that allows to estimate  $\|f(A)\|$  in terms of the  $\mathbf{L}^p(\mathbb{R}; X)$ -Fourier multiplier norm of  $f(\cdot \pm i\omega)$ . If  $X$  is a Hilbert space this yields new proofs of important results of McIntosh and Boyadzhiev-de Laubenfels. If  $X$  is a UMD space, one obtains a bounded  $H_1^\infty$ -calculus of  $A$  on horizontal strips. Analogous results for sectorial and parabola-type operators follow. Finally we prove that each generator of a cosine function has bounded  $H^\infty$ -calculus on sectors.

## 1. INTRODUCTION

The name “transference principle” was introduced in [17] by COIFMAN and WEISS to unify certain results from harmonic analysis. The original setting is in terms of a locally compact abelian group acting boundedly on some space  $X = \mathbf{L}^p(\Omega, \mu)$ , but it was observed by BERKSON, GILLESPIE and MUHLY in [10] that the restriction to  $\mathbf{L}^p$ -spaces was unnecessary and one could in fact take as  $X$  any Banach space. Their result was later generalised by BERKSON, PALUZYŃSKI and WEISS to so-called *transference couples* [11]. For the group of real numbers, the “classical” transference principle has the following form (see [10, Theorem 2.8] or [28, Theorem 10.5] for proofs).

**Theorem 1.1. (Coifman-Weiss [17], Berkson–Gillespie–Muhly [10])**

Let  $-iA$  generate a  $C_0$ -group  $U$  on a Banach space  $X$ , such that  $\|U(t)\| \leq M$  for all  $t \in \mathbb{R}$ . Then

$$\left\| \int_{\mathbb{R}} U(s)x \mu(ds) \right\| \leq M^2 \|L_\mu\|_{\mathcal{L}(\mathbf{L}^p(\mathbb{R}; X))} \|x\|_X$$

for all  $x \in X$ ,  $\mu \in \mathbf{M}(\mathbb{R})$ ,  $1 \leq p < \infty$ . Here,  $L_\mu$  denotes the convolution operator  $L_\mu := (f \mapsto \mu * f)$ .

So “transference” means that certain averages over the representation of the group can be estimated by the norm of the associated convolution operator. In this manner one can for example prove that restrictions to  $\mathbb{Z}$  of Fourier multipliers on  $\mathbb{R}$  are Fourier multipliers on  $\mathbb{T}$ , but there are plenty of other examples and applications of transference, see [18, 6, 7, 8, 34, 2, 9, 3, 5].

Although the transference result in itself is valid in any Banach space, to be applicable one usually needs special conditions. The by far most useful of these seems to be the so-called UMD-property (see section 3 below). It were CLÉMENT and PRÜSS in [15] who observed the usefulness of the vector-valued (in particular:

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UMD-valued) transference principle for treating problems in evolution equations. The link is via the group of imaginary powers of a sectorial operator, and is based on the 1987 paper [20] of DORE and VENNI which established the relation between bounded imaginary powers and the maximal regularity property of a sectorial operator. A few years later, when MCINTOSH's notion of a "bounded  $H^\infty$ -calculus" (from [30]) had become adopted by the evolution equations community, HIEBER and PRÜSS used Theorem 1.1 to derive the following result.

**Theorem 1.2. (Hieber–Prüss [27])**

*Let  $-iA$  generate a bounded  $C_0$ -groups on a UMD space  $X$ . Then for every  $\theta \in (0, \pi/2)$  the operator  $A$  has a bounded  $H^\infty(\Sigma_\varphi)$ -calculus.*

(The symbol  $\Sigma_\varphi$  denotes the double sector of angle  $\varphi$ , see below.) For proofs see also [28, Theorem 10.7] and Remark 3.8 below.

Around 2000 the notion of  $R$ -boundedness was fully developed and its importance for operator-valued Fourier multiplier theorems and the maximal regularity problem had been recognised by many authors; key steps were the paper [14] and, of course, WEIS' paper [36]. Using the notion of  $R$ -boundedness, CLÉMENT and PRÜSS extended in [16] the transference principle and Theorem 1.2 to the operator-valued setting and gave applications to the maximal regularity problem. See [36, 16, 19, 28] for more information.

Despite the wide range of applications, it is clear that the confinement to bounded representations of groups is a major restriction; however, it seems that no transference result for unbounded groups is available yet in the literature. In this article we shall fill this gap (Theorem 3.2). Moreover, we shall apply the transference principle to obtain a generalisation of the Hieber–Prüss theorem to unbounded groups (Theorem 4.6), and to obtain boundedness results for certain functional calculi of unbounded operators (Theorem 5.1); these results are the UMD-analogues of the Hilbert space theorems of MCINTOSH [30] and of BOYADZHIEV and DE LAUBENFELS [12]. Other applications are to generators of cosine functions on UMD spaces (Theorem 5.3, Corollary 5.6). Our results can also be used to obtain new proofs for important results of FATTORINI [1, Theorem 3.16.7], DORE and VENNI [20] and MONNIAUX [31], but this is done in a different paper [24].

The paper is organised as follows. In Section 2 we provide without proofs the necessary material on functional calculus. In Section 3 we prove the main transference result (Theorem 3.2). Then we introduce UMD spaces and use BOURGAI'S UMD-valued multiplier result of Mihlin type to derive our main result on functional calculus (Theorem 3.6). In Section 4 we discuss some examples, in a particular we describe a class of functions  $f$  such that the operator  $f(A)$  is explicitly given by a principal value integral (Theorem 4.4). We also give the generalisation of the Hieber–Prüss result to unbounded groups. Section 5 is devoted to the consequences our main results have in the context of sectorial and parabola-type operators.

*Notation*

We usually consider (unbounded) closed operators  $A, B$  on a Banach space  $X$ . By  $\mathcal{L}(X)$  we denote the set of all *bounded* (fully-defined) operators on  $X$ . The domain and the range of a general operator  $A$  are denoted by  $\mathcal{D}(A)$  and  $\mathcal{R}(A)$ , respectively. Its resolvent is  $R(\lambda, A) = (\lambda - A)^{-1}$ , and  $\varrho(A)$  denotes the set of  $\lambda \in \mathbb{C}$  where  $R(\lambda, A) \in \mathcal{L}(X)$ . Its complement  $\sigma(A) = \mathbb{C} \setminus \varrho(A)$  is the *spectrum*. For given  $\omega > 0$

we define

$$St_\omega := \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \omega\}$$

to be the horizontal strip of height  $2\omega$ . If  $\omega = 0$  we define  $St_0 := \mathbb{R}$ . Accordingly

$$\Pi_\omega := \{z^2 \mid z \in St_\omega\}$$

is the horizontal parabola, symmetric about  $\mathbb{R}$  and open to the right. Furthermore, for  $\omega \in [0, \pi)$

$$S_\omega := \{e^z \mid z \in St_\omega\} = \begin{cases} \{z \neq 0 \mid |\arg z| < \omega\}, & \omega \in (0, \pi] \\ (0, \infty), & \omega = 0 \end{cases}$$

is the horizontal sector of angle  $2\omega$ , symmetric about the positive real axis. For  $\omega \in [0, \pi/2)$

$$\Sigma_\omega := S_\omega \cup -S_\omega$$

is the horizontal double sector.

For each open subset  $\Omega \subset \mathbb{C}$  we denote by  $H^\infty(\Omega)$  the Banach algebra of bounded holomorphic functions on  $\Omega$ . If  $\Omega$  is an arbitrary locally compact space, then the set of complex Borel measures on  $\Omega$  is denoted by  $\mathbf{M}(\Omega)$ . The *Fourier transform* of a tempered distribution  $\Phi$  on  $\mathbb{R}$  is denoted by  $\mathcal{F}(\Phi)$  or  $\hat{\Phi}$ . We often write  $s$  and  $t$  (in the Fourier image) to denote the real coordinate, e.g.  $\sin t/t$  denotes the *function*  $t \mapsto \sin t/t$ .

Let  $X$  be a Banach space. For a finite measure  $\nu \in \mathbf{M}(\mathbb{R})$  we denote by

$$L_\nu := (f \rightarrow f * \nu) : \mathbf{L}^p(\mathbb{R}; X) \rightarrow \mathbf{L}^p(\mathbb{R}; X)$$

the convolution operator on the  $X$ -valued  $\mathbf{L}^p$ -space.

## 2. FUNCTIONAL CALCULUS PRELIMS

In this section we provide necessary facts on functional calculus. Most of the results are well-known for a long time, cp. [33], [27] and the comments in [25].

Let  $-iA$  be the generator of a  $C_0$ -group  $(U(s))_{s \in \mathbb{R}}$  on a Banach space  $X$ . It is an elementary fact that there exist constants  $M \geq 1, \omega \geq 0$  such that  $\|U(s)\| \leq M e^{\omega|s|}$ ,  $s \in \mathbb{R}$ . The infimum of all such  $\omega \geq 0$  is called the *group type* of  $U$  and is (here) denoted by  $\theta(U)$ . Abstract semigroup theory yields that the resolvent of  $A$  satisfies the estimate

$$\|R(\lambda, A)\| \leq \frac{M}{|\operatorname{Im} \lambda| - \omega} \quad (|\operatorname{Im} \lambda| > \omega),$$

hence  $A$  is a so-called strong strip type operator of height  $\omega_{sst}(A) \leq \omega$ , as defined in [25, Section 4.1] (see also [26, Definition 6.1]). There is a natural holomorphic functional calculus associated with operators of such a kind: in a first step one uses the Cauchy formula to define

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A) dz,$$

where  $f$  is a holomorphic function on a strip in the class  $\mathcal{E}(\theta)$  defined by

$$\mathcal{E}(\theta) := \{f \in H^\infty(St_\theta) \mid f(z) = O(|z|^{-2}) (\operatorname{Re} z \rightarrow \infty)\},$$

for some  $\theta > \omega$ . The contour  $\Gamma$  is the positively oriented boundary of a smaller strip  $St_{\omega'}$ ,  $\omega' \in (\omega, \theta)$  being arbitrary. This yields an algebra homomorphism of the algebra  $\mathcal{E}(\theta)$  into the algebra of bounded operators on  $X$ . In a second step, by so-called regularisation, one defines  $f(A)$  for a much wider class of functions:

$$f(A) := e(A)^{-1}(ef)(A)$$

where  $e \in \mathcal{E}(\theta)$  is such that also  $ef \in \mathcal{E}(\theta)$  and  $e(A)$  is injective. The function  $e$  is called a *regulariser* for  $f$ , and the definition of  $f(A)$  is independent of the chosen regulariser. For example, if  $f \in H^\infty(St_\theta)$ , then  $f$  is regularisable by any function  $e(z) = (\lambda - z)^{-2}$ , where  $|\operatorname{Im} \lambda| > \theta$ . Details of the construction as well as a listing of all the formal properties of the so constructed functional calculus can be found in [25, Chapter 1 and 4].

Of particular importance in the theory of functional calculus is the so-called convergence lemma. It goes back to MCINTOSH [30] in the sectorial case; for a proof see or [24] or [25, Prop. 5.1.4].

**Proposition 2.1. (Convergence Lemma)** *Let  $A$  be a strong strip-type operator on the Banach space  $X$ , with dense domain. Let  $\theta > \omega_{st}(A)$ , and let  $(f_\iota)_{\iota \in \mathcal{J}}$  be a net of holomorphic functions on the strip  $St_\theta$ , satisfying*

- 1)  $\sup\{|f_\iota(z)| \mid z \in St_\theta, \iota \in \mathcal{J}\} < \infty$ ;
- 2)  $f_\iota(z) \rightarrow f(z)$  for every  $z \in St_\theta$ ;
- 3)  $\sup_\iota \|f_\iota(A)\| < \infty$ .

*Then  $f(A) \in \mathcal{L}(X)$  and  $f_\iota(A) \rightarrow f(A)$  strongly.*

In the case that  $-iA$  generates a  $C_0$ -group there is a convenient tool to identify functions  $f$  such that  $f(A)$  is a bounded operator. First of all, consider for  $s \in \mathbb{R}$  the function  $e^{-isz}$ , which is bounded on every horizontal strip. We clearly expect  $e^{-isz}(A) = U(s)$  for all  $s \in \mathbb{R}$ . More generally, let  $\omega \geq 0$  be fixed such that  $\|U(s)\| \leq Me^{\omega|s|}$  for some  $M \geq 1$  and all  $s \in \mathbb{R}$ , and let  $\mu$  be a (complex) Borel measure on  $\mathbb{R}$  satisfying

$$\|\mu\|_{\mathbf{M}_\omega} := \int_{\mathbb{R}} e^{\omega|s|} |\mu|(ds) < \infty. \quad (1)$$

Then one can set

$$T_\mu x := \int_{\mathbb{R}} U(s)x \mu(ds) \quad (x \in X).$$

Clearly, the set  $\mathbf{M}_\omega(\mathbb{R}) := \{\mu \mid (1) \text{ holds}\}$  is a Banach algebra with respect to convolution, and the map  $(\mu \mapsto T_\mu) : \mathbf{M}_\omega(\mathbb{R}) \rightarrow \mathcal{L}(X)$  is a homomorphism of algebras, called the *Phillips calculus*. Of course we expect  $T_\mu = f(A)$  where  $f$  is the *Fourier-Stieltjes transform*

$$f(z) = \widehat{\mu}(z) := \int_{\mathbb{R}} e^{-isz} \mu(ds) \quad (z \in St_\theta)$$

of  $\mu$ . Note that  $\widehat{\mu} \in H^\infty(St_\omega) \cap \mathbf{C}^b(\overline{St_\omega})$ . Here is the precise result, a proof of which is in [24], see also [27, Section 2].

**Lemma 2.2.** *Let  $X$ ,  $A$ , and  $U$  be as above, and let  $\theta > \omega$ .*

- a) *Each function  $f \in \mathcal{E}(\theta)$  arises as a Fourier–Stieltjes transform, namely*

$$f = \widehat{g} \quad \text{with} \quad g(s) := \frac{1}{2\pi} \int_{\mathbb{R}} f(t)e^{ist} dt \quad (s \in \mathbb{R}).$$

*One has  $g \in \mathbf{C}_0(\mathbb{R})$  and  $\int |g(s)| e^{\alpha|s|} ds < \infty$  for all  $\alpha \in [0, \theta)$ ; in particular, one has  $g(t)dt \in \mathbf{M}_\omega(\mathbb{R})$ .*

- b) *Let  $\mu \in \mathbf{M}_\omega(\mathbb{R})$ , and suppose that  $f := \widehat{\mu}$  extends to a holomorphic function on  $St_\theta$  such that  $f(A)$  is defined. Then  $f(A) = T_\mu \in \mathcal{L}(X)$  and*

$$\sup_{t \in \mathbb{R}} \|f(t + A)\| \leq M \|\mu\|_{\mathbf{M}_\omega}.$$

The formulation of part b) in the above proposition is due to the fact that in our original setup of the functional calculus functions had to be defined on strips strictly larger than the spectral strip. By Lemma 2.2 we can extend the original definition and write

$$f(A)x := T_\mu x := \int_{\mathbb{R}} U(t)x \mu(dt) \quad (x \in X)$$

whenever  $f = \widehat{\mu}, \mu \in \mathbf{M}_\omega(\mathbb{R})$ . This will induce also a compatible extension of the unbounded functional calculus, see [25, Proposition 1.2.7].

However, as far as *bounded* operators are concerned, one cannot go beyond Fourier transforms of measures  $\mu \in \mathbf{M}_\omega(\mathbb{R})$  in general. Indeed, if  $f(A)$  is bounded in the special case of  $-iA = d/dt$  generating the shift group on

$$X := \mathbf{L}_\omega^1(\mathbb{R}) := \{f \in \mathbf{L}_{\text{loc}}^1(\mathbb{R}) \mid \int_{\mathbb{R}} f(t)e^{\omega|t|} dt < \infty\},$$

then actually  $f = \widehat{\mu}$  for some  $\mu \in \mathbf{M}_\omega(\mathbb{R})$  (see for example [24, Proposition 2.3]). One can put this remark in the form of a transference principle.

**Proposition 2.3. (Transference Principle,  $\mathbf{L}^1$ -version)**

Let  $-iA$  generate  $C_0$ -group  $U$  on a Banach space  $X$ , and let  $M \geq 1, \omega \geq 0$  be such that  $e^{-\omega|t|} \|U(t)\| \leq M$  for all  $t \in \mathbb{R}$ . Then

$$\|f(A)\|_{\mathcal{L}(X)} \leq M \|f(id/dt)\|_{\mathcal{L}(\mathbf{L}_\omega^1(\mathbb{R}))}$$

for all  $f = \widehat{\mu}, \mu \in \mathbf{M}_\omega(\mathbb{R})$ .

*Proof.* Let  $\mu \in \mathbf{M}_\omega(\mathbb{R})$ , define  $f := \widehat{\mu}$ . If  $A_0 := id/dt$  on  $\mathbf{L}_\omega^1(\mathbb{R})$ , it is easily seen that (1)  $f(A_0)$  is convolution with the measure  $\mu^\sim$ , defined by  $\mu^\sim(B) = \mu(-B)$ , and that (2) the norm of this convolution operator is exactly  $\|\mu\|_{\mathbf{M}_\omega}$  (see [24, (2.2)]). But it is trivial that

$$\|f(A)\| = \|T_\mu\| \leq M \|\mu\|_{\mathbf{M}_\omega}$$

and so we are done. □

Let us mention that although this first example of a transference principle is fairly elementary, it has important consequences. See e.g. [4] for the case  $\omega = 0$ , where the authors give also a different proof.

It is known for a long time that if  $-iA$  generates a group on a UMD space  $X$ ,  $f(A)$  is bounded even for functions  $f$  arising from certain principal value distributions (see also Section 4 below.) In fact, this is the core of the results of Dore–Venni, Monniaux and Fattorini, but this was fully brought to light only recently in [24]. (We are oversimplifying here as a more detailed discussion would lead us too far astray.) In the following we will extend the results from [24] towards a full bounded functional calculus.

### 3. THE TRANSFERENCE PRINCIPLE

In the effort to obtain an analogue for unbounded groups of the classical Coifman–Weiss transference principle (Theorem 1.1) a first progress was made recently in [24, Theorem 3.1]. We reproduce the result here, because its proof is short and instructive.

**Theorem 3.1. (Transference Principle, fixed compact support [24])**

Let  $p \in [1, \infty)$ , and let  $U$  be a  $C_0$ -group on a Banach space  $X$ . Define  $M :=$

$\sup_{s \in [-2, 2]} \|U(s)\|$ . Then

$$\left\| \int_{[-1, 1]} U(s)x \mu(ds) \right\| \leq 2^{1/p} M^2 \|L_\mu\|_{\mathcal{L}(\mathbf{L}^p(\mathbb{R}; X))} \|x\| \quad (x \in X)$$

for all  $\mu \in \mathbf{M}[-1, 1]$ .

*Proof.* Let  $|t| \leq 1$ . Then we write

$$T_\mu x = \int_{-1}^1 U(s)x \mu(ds) = U(t) \int_{-1}^1 U(s-t)x \mu(ds) = U(t)(f * \mu)(t)$$

where  $f(s) = \mathbf{1}_{[-2, 2]}(s) U(-s)x$ . Hence

$$\|T_\mu x\| = \left\| \frac{1}{2} \int_{-1}^1 U(t)(f * \mu)(t) dt \right\| \leq 2^{\frac{1}{p}-1} M \|f * \mu\|_{\mathbf{L}^p(\mathbb{R}; X)} \leq 2^{\frac{1}{p}} M^2 \|L_\mu\| \|x\|,$$

where  $\|L_\mu\| = \|L_\mu\|_{\mathcal{L}(\mathbf{L}^p(\mathbb{R}; X))}$ .  $\square$

In Theorem 3.1 the actual growth type of the group is irrelevant since only measures  $\mu$  with compact support are considered. If we allow unbounded support, we have to modify the proof. However, the strategy will be the same: we first convolve the (slightly modified) measure  $\mu$  with some function in  $\mathbf{L}^p$ , then we integrate (against a “test function”) and in the end obtain the operator  $T_\mu$ . Given  $\mu \in \mathbf{M}_\omega(\mathbb{R})$  we define the finite measure  $\mu_\omega$  by

$$\mu_\omega(ds) = \cosh(\omega s) \mu(ds).$$

Then the result is as follows.

**Theorem 3.2. (Transference Principle)**

Let  $0 \leq \omega_0 < \omega$ , and  $p \in [1, \infty)$ . Then there is a constant  $C = C(p, \omega_0, \omega)$  such that the following holds: If  $U$  is a  $C_0$ -group on a Banach space  $X$  such that  $\|U(s)\| \leq M \cosh(\omega_0 s)$  for all  $s \in \mathbb{R}$  and some  $M \geq 1$ , then

$$\left\| \int_{\mathbb{R}} U(s)x \mu(ds) \right\| \leq CM^2 \|L_{\mu_\omega}\|_{\mathcal{L}(\mathbf{L}^p(\mathbb{R}; X))} \|x\| \quad (x \in X)$$

for all  $\mu \in \mathbf{M}_\omega(\mathbb{R})$ .

*Proof.* Fix  $\alpha > \omega$  and  $x \in X$ , and consider the function  $f(s) := [\cosh(\alpha s)]^{-1} U(-s)x$ ,  $s \in \mathbb{R}$ . Clearly  $f \in \mathbf{L}^p(\mathbb{R}; X)$  and

$$\|f\|_{\mathbf{L}^p(\mathbb{R}; X)} \leq M c_1 \|x\|$$

with  $c_1 := \|\cosh(\omega_0 s) / \cosh(\alpha s)\|_{\mathbf{L}^p(\mathbb{R})}$ . Hence

$$\|f * \mu_\omega\|_{\mathbf{L}^p} = \|L_{\mu_\omega} f\|_{\mathbf{L}^p} \leq M \|L_{\mu_\omega}\|_{\mathcal{L}(\mathbf{L}^p(\mathbb{R}; X))} c_1 \|x\|.$$

Note that for every  $t \in \mathbb{R}$

$$\begin{aligned} (f * \mu_\omega)(t) &= \int_{\mathbb{R}} U(s-t)x \frac{\cosh(\omega s)}{\cosh(\alpha(s-t))} \mu(ds) \\ &= U(-t) \int_{\mathbb{R}} U(s)x \frac{\cosh(\omega s)}{\cosh(\alpha(s-t))} \mu(ds). \end{aligned}$$

Let  $\varphi$  be a scalar function such that  $\varphi(s) = O(\cosh(\omega s)^{-1})$  as  $|s| \rightarrow \infty$ . Then

$$\begin{aligned} T &:= \int_{\mathbb{R}} \varphi(t) U(t) [f * \mu_\omega](t) dt \\ &= \int_{\mathbb{R}} [\varphi * \cosh(\alpha \cdot)^{-1}](s) \cosh(\omega s) U(s)x \mu(ds) \end{aligned}$$

by Fubini's theorem. If we can choose  $\varphi$  such that

$$[\varphi * \cosh(\alpha \cdot)^{-1}](s) = [\cosh(\omega s)]^{-1} \quad (s \in \mathbb{R}) \quad (2)$$

then  $T = T_\mu$  and

$$\|T_\mu\| = \|T\| \leq M \|\varphi \cosh(\omega_0 \cdot)\|_{\mathbf{L}^{p'}} M \|L_\mu\|_{\mathcal{L}(\mathbf{L}^p(\mathbb{R}; X))} c_1 \|x\|.$$

To determine such a  $\varphi$  we take Fourier transforms on both sides of (2). It is known that the function  $\cosh^{-1}$  is almost its own Fourier transform. More precisely, one has

$$\int_{\mathbb{R}} \frac{e^{-isz}}{\cosh(\omega s)} ds = \frac{\pi/\omega}{\cosh((\pi/2\omega)z)} \quad (|\operatorname{Im} z| < \omega)$$

(see [35, p.81] for a proof). Hence we look for a function  $\varphi$  that satisfies

$$\widehat{\varphi}(z) \frac{\pi/\alpha}{\cosh((\pi/2\alpha)z)} = \frac{\pi/\omega}{\cosh((\pi/2\omega)z)}$$

that is

$$\varphi(t) = \frac{\alpha}{\omega} \mathcal{F}^{-1} \left( \frac{\cosh((\pi/2\alpha) \cdot)}{\cosh((\pi/2\omega) \cdot)} \right) (t) = \frac{2\alpha}{\pi} \cos \left( \frac{\pi\alpha}{2\omega} \right) \frac{\cosh(\omega t)}{\cos(\pi\omega/\alpha) + \cosh(2\omega t)}$$

(computed from the second formula of [32, p.36]). For example, taking  $\alpha = 2\omega$  the function

$$\varphi(t) = \frac{-4\omega}{\pi} \frac{\cosh(\omega t)}{\cosh(2\omega t)} \quad (t \in \mathbb{R})$$

will do. □

**Remark 3.3.** It is worthwhile to compare Theorem 3.2 with Theorem 1.1. In the proof of the former it is essential that  $\omega$  is strictly larger than  $\omega_0$ , the exponential growth type of the group. This matches the experience that bounded groups behave much better than general ones. We do not expect the conclusion of Theorem 3.2 to hold for  $\omega = \omega_0$ .

As in the classical case, one can phrase the transference principle in terms of norms of Fourier multipliers. Namely, one has

$$2\mu_\omega(ds) = 2 \cosh(\omega s) \mu(ds) = e^{\omega s} \mu(ds) + e^{-\omega s} \mu(ds),$$

and since

$$\mathcal{F}(e^{\omega \cdot} \mu) = \widehat{\mu}(\cdot - i\omega) \quad \text{and} \quad \mathcal{F}(e^{-\omega \cdot} \mu) = \widehat{\mu}(\cdot + i\omega),$$

the convolution operator  $L_{\mu_\omega}$  from above is the average of the two Fourier multiplier operators with symbols  $\widehat{\mu}(\cdot \pm i\omega)$ . Denoting the space of bounded  $\mathbf{L}^p(\mathbb{R}; X)$ -Fourier multipliers with  $\mathcal{M}_p(X)$ , we obtain the following.

**Corollary 3.4.** *Fix  $0 \leq \omega_0 < \omega$  and  $p \in [1, \infty)$ . Then there is a constant  $C = C(p, \omega_0, \omega)$  such that the following holds: If  $U$  is a  $C_0$ -group on a Banach space  $X$  such that  $\|U(s)\| \leq M \cosh(\omega_0 s)$  for all  $s \in \mathbb{R}$  and some  $M \geq 1$ , then*

$$\left\| \int_{\mathbb{R}} U(s) x \mu(ds) \right\| \leq CM^2 \left( \|\widehat{\mu}(\cdot + i\omega)\|_{\mathcal{M}_p(X)} + \|\widehat{\mu}(\cdot - i\omega)\|_{\mathcal{M}_p(X)} \right) \|x\|$$

for all  $x \in X$  and all  $\mu \in \mathbf{M}_\omega(\mathbb{R})$ .

As in the classical case, the transference principle shows its full strength in a situation when one actually knows something about  $\mathbf{L}^p(\mathbb{R}; X)$ -Fourier multipliers. This is the case when the Banach space has the so-called UMD property.

A Banach space  $X$  has the UMD property if every  $X$ -valued  $\mathbf{L}^p$ -martingale has unconditional differences. This notion was introduced by BURKHOLDER [13] and it

is independent of  $p \in (1, \infty)$ . BOURGAIN showed that the UMD property is in fact equivalent to the boundedness of the Hilbert transform on  $\mathbf{L}^2(\mathbb{R}; X)$ . To make this explicit, consider the truncated Hilbert transform

$$\mathcal{H}_\varepsilon f(s) := \int_{|t| \geq \varepsilon} f(s-t) \frac{dt}{t} \quad (f \in \mathbf{L}^2(\mathbb{R}; X)).$$

Then  $X$  is UMD if and only if  $(T_\varepsilon)_\varepsilon$  is uniformly bounded in  $\mathcal{L}(\mathbf{L}^2(\mathbb{R}; X))$  if and only if  $(\mathcal{H}_\varepsilon)_\varepsilon$  converges as  $\varepsilon \searrow 0$  strongly in  $\mathcal{L}(\mathbf{L}^2(\mathbb{R}; X))$  to a bounded operator  $\mathcal{H}$ . The operator  $\mathcal{H}$  is called the Hilbert transform. On UMD spaces we have the following Mihklin-type multiplier theorem [37].

**Lemma 3.5. (Mikhlin, UMD-valued [37])**

Let  $X$  be a UMD space and  $p \in (1, \infty)$ . Then there is a constant  $C_p$  such that every  $m \in \mathbf{C}^1(\mathbb{R} \setminus \{0\})$  with

$$c_m := \sup_{t \in \mathbb{R} \setminus \{0\}} |m(t)| + \sup_{t \in \mathbb{R} \setminus \{0\}} |tm'(t)| < \infty.$$

is a bounded  $\mathbf{L}^p(\mathbb{R}; X)$ -multiplier such that  $\|m\|_{\mathcal{M}_p(X)} \leq C_p c_m$ .

Using this result, we can now state and prove the main result on functional calculus. For  $\theta > 0$  define

$$H_1^\infty(St_\theta) := \{f \in \mathcal{O}(St_\theta) \mid f, zf' \in H^\infty(St_\theta)\}.$$

This set is a Banach algebra with the norm

$$\|f\|_{H_1^\infty} := \sup_{z \in St_\theta} |f(z)| + |zf'(z)|.$$

Every elementary rational function  $(\lambda - z)^{-1}$ ,  $|\operatorname{Im} \lambda| > \theta$  belongs to  $H_1^\infty(St_\theta)$ .

**Theorem 3.6. (Functional Calculus)**

Let  $X$  be a UMD space, and let  $-iA$  be the generator of a strongly continuous group  $U = (U(s))_{s \in \mathbb{R}}$  on  $X$ . Let  $\theta > \theta(U)$ . Then  $f(A+r)$  is bounded for every  $f \in H_1^\infty(St_\theta)$  and every  $r \in \mathbb{R}$ , and there is a constant  $c > 0$  such that

$$\|f(A+r)\| \leq c (\|f\|_\infty + \|zf'(z)\|_\infty) \quad (f \in H_1^\infty(St_\theta), r \in \mathbb{R}).$$

*Proof.* Let  $M \geq 1$  and  $\omega_0 \in [0, \theta)$  such that  $\|U(s)\| \leq M \cosh(\omega_0 s)$ ,  $s \in \mathbb{R}$ , and choose  $\omega \in (\omega_0, \theta)$ . In a first step, take  $f \in H_1^\infty(St_\theta) \cap \mathcal{E}(St_\theta)$ . Then by Lemma 2.2 there is a function  $g$  on  $\mathbb{R}$  such that  $g(t)dt \in \mathbf{M}(\mathbb{R})$  and  $\hat{g} = f$ . Fixing  $x \in X$  and applying Corollary 3.4 yields

$$\begin{aligned} \|f(A+r)x\| &= \left\| \int_{\mathbb{R}} g(s) e^{-isr} U(s)x \, ds \right\| \\ &\leq CM^2 \|x\| \left( \|f(\cdot + i\omega)\|_{\mathcal{M}_2(X)} + \|f(\cdot - i\omega)\|_{\mathcal{M}_2(X)} \right) \end{aligned}$$

which by Lemma 3.5 can be estimated by

$$\begin{aligned} \dots &\leq C'M^2 \|x\| \left( \|f\|_\infty + \sup_{t \in \mathbb{R}} |tf'(t + i\omega)| + |tf'(t - i\omega)| \right) \\ &\leq C'M^2 \|x\| \|f\|_{H_1^\infty(St_\theta)}. \end{aligned}$$

To complete the proof of the theorem, we employ the Convergence Lemma (Proposition 2.1). Let  $\tau_n(z) := in(in - z)^{-1}$  for  $n \in \mathbb{N}$  large. Then  $\tau_n \in H_1^\infty$ ,  $\sup_n \|\tau_n\|_{H_1^\infty} < \infty$  and  $\tau_n \rightarrow 1$  uniformly on compacts. Let  $f \in H_1^\infty$  be arbitrary and define  $f_n := f\tau_n^2$ . Then  $k := \sup_n \|f\tau_n\|_{H_1^\infty} < \infty$ , in particular:

$\sup_n \|f\tau_n\|_\infty < \infty$ , and  $f\tau_n \rightarrow f$  uniformly on compacts. Clearly  $f_n \in \mathcal{E}(St_\theta)$ , so we know already that

$$\|f_n(A+r)\| \leq c' \|f_n\|_{H_1^\infty(St_\theta)} \leq c'k$$

independent of  $n \in \mathbb{N}$  and  $r \in \mathbb{R}$ . Applying the Convergence Lemma yields that  $f(A+r) \in \mathcal{L}(X)$  and

$$\|f(A+r)\| \leq C'M^2 \left( \limsup_n \|\tau_n^2\|_{H_1^\infty} \right) \|f\|_{H_1^\infty}$$

for all  $r \in \mathbb{R}$ . □

If  $X$  happens to be a Hilbert space, one obtains a much better result. In fact, instead of Lemma 3.5 one can use Plancherel's theorem and estimate  $\|L_\mu\| \leq \frac{1}{2} \|\widehat{\mu}(\cdot + i\omega)\|_\infty + \frac{1}{2} \|\widehat{\mu}(\cdot - i\omega)\|_\infty \leq \|\widehat{\mu}\|_{H^\infty}$ . By using the Convergence Lemma as in the previous proof, this leads to the following.

**Corollary 3.7. (Boyadzhiev–de Laubenfels [12])**

*Let  $-iA$  be the generator of a  $C_0$ -group  $U$  on a Hilbert space  $H$ . Then for every  $\theta > \theta(U)$ , the natural  $H^\infty(St_\theta)$ -calculus is bounded.*

This theorem was originally proved in [12], but subsequently reproved in [21] and [23], cf. also [25, Section 7.2].

**Remark 3.8.** The proof of Theorem 3.6 carries over to a proof of the Hieber–Prüss Theorem 1.2; one has to use Theorem 1.1 instead of Theorem 3.2, and Lemma 4.5 below.

4. SOME CLASSES OF EXAMPLES

We are going to discuss some classes of functions  $f \in H_1^\infty$ . The first avoids involving derivatives.

**Lemma 4.1.** *Let  $\omega > 0$  and let  $f \in H^\infty(St_\omega)$ . If there exists  $a, b \in \mathbb{C}$  such that  $f(z) - a = O(z^{-1})$  as  $\operatorname{Re} z \rightarrow +\infty$  and  $f(z) - b = O(z^{-1})$  as  $\operatorname{Re} z \rightarrow -\infty$ , then  $f \in H_1^\infty(St_\theta)$  for every  $\theta \in (0, \omega)$ .*

*Proof.* Define  $g(z) = zf(z) - az$ . This is bounded on the half-strip ( $\operatorname{Re} z \geq 0, |\operatorname{Im} z| < \omega$ ). Cauchy's formula yields

$$g'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{g(w)}{(w-z)^2} dw$$

where  $\Gamma$  is the positively oriented boundary of a right half-strip within the original one and  $z$  is within this smaller half-strip. Consequently,  $g'$  is uniformly bounded on every even smaller half-strip. But  $g'(z) = f(z) + zf'(z) - a$ , and so  $zf'(z)$  is bounded on that half-strip. Analogously,  $zf'(z)$  is bounded on left half-strips, and so we conclude that  $zf'(z)$  is bounded on whole strips  $St_\theta, \theta \in (0, \omega)$ . □

We now turn to a class of examples where one actually has a representation of  $f(A)$  as a principal value integral. For an even function  $g \in \mathbf{L}^1(-1, 1)$  (i.e.,  $g(t) = g(-t)$ ) we define the distribution  $\operatorname{PV} - g(s)/s$  by the formula

$$\left\langle \operatorname{PV} - \frac{g(s)}{s}, \varphi \right\rangle := \lim_{\varepsilon \searrow 0} \int_{\varepsilon < |s| < 1} g(s)\varphi(s) \frac{ds}{s} = \int_0^1 g(s) \frac{\varphi(s) - \varphi(-s)}{s} ds$$

for  $\varphi \in \mathcal{D}(\mathbb{R})$ . Then it is clear that

$$\left| \left\langle \operatorname{PV} - \frac{g(s)}{s}, \varphi \right\rangle \right| \leq \|g\|_{\mathbf{L}^1(-1,1)} \|\varphi'\|_\infty$$

whence  $PV - g(t)/t$  is in fact a distribution of first order. The proof of the following lemma is easy (see [24]).

**Lemma 4.2.** *Let  $g \in \mathbf{L}^1(-1,1)$  be even and define  $h := PV - g(t)/t$  as above. Then the following assertions hold.*

- a)  $h$  is an odd distribution.
- b) Its Fourier transform is

$$\widehat{h}(z) = PV - \int_{-1}^1 g(t)e^{-isz} \frac{ds}{s} = (-2i) \int_0^1 \frac{\sin(sz)}{s} g(s) ds \quad (z \in \mathbb{C})$$

- c) One has  $\frac{d}{dz} \widehat{h}(z) = (-i) \widehat{g}(z)$ ,  $z \in \mathbb{C}$ , and  $\widehat{h}(0) = 0$ .

In [24] we considered even functions  $g \in \mathbf{L}^1[1,1]$  such that, for some  $c \in \mathbb{C}$ ,

$$\int_0^1 \left| \frac{g(s) - c}{s} \right| ds < \infty;$$

this reduces the problem to the case that  $g$  is constant. Here we can give a generalisation where we merely assume that  $g$  has bounded variation.

**Lemma 4.3.** *Let  $g \in \mathbf{BV}[-1,1]$  be an even function and define  $f := \mathcal{F}(PV - g(s)/s)$ . Then  $f \in H_1^\infty(St_\theta)$ , for every  $\theta > 0$ . Moreover, for each  $\theta > 0$  there is a constant  $c_\theta$  such that*

$$\|f\|_{H_1^\infty(St_\theta)} \leq c_\theta (\text{Var}_{[0,1]}(g) + g(1)) \quad (g \in \mathbf{BV}[0,1], f = \mathcal{F}(PV - g(s)/s))$$

*Proof.* We first estimate  $|zf'(z)|$  and write

$$\begin{aligned} zf'(z) &= (-i) \widehat{g}(z) = (-2i) \int_0^1 z \cos(sz) g(s) ds = (-2i) g(s) d(\sin(sz)) \\ &= (-2i) \left( g(1) \sin(z) - \int_0^1 \sin(sz) dg(s) \right). \end{aligned}$$

This yields

$$|zf'(z)| \leq 2e^\theta (\text{Var}_{[0,1]}(g) + g(1)) \quad (z \in St_\theta).$$

To estimate  $|f(z)|$  itself, we write

$$\begin{aligned} f(z) &= (-2i) \int_0^1 \frac{\sin(sz)}{s} g(s) ds = (-2i) \int_0^1 g(s) d \left( \int_0^s \frac{\sin(rz)}{r} dr \right) \\ &= (-2i) \left( g(1) \int_0^1 \frac{\sin(rz)}{r} dr - \int_0^1 \int_0^s \frac{\sin(rz)}{r} dr dg(s) \right) \end{aligned}$$

Hence  $|f(z)| \leq 2c'_\theta (\text{Var}_{[0,1]}(g) + g(1))$ , where  $c'_\theta$  is the supremum norm of the function  $\int_0^1 \sin(sz) ds/s$  on the strip  $St_\theta$ ; this is easily seen to be finite, cp. [24, Lemma 3.3].  $\square$

If  $g, f$  are as before, then by Theorem 3.6,  $f(A)$  is a bounded operator whenever  $-iA$  generates a  $C_0$ -group on a UMD space  $X$ . However, we can say (a little) more.

**Theorem 4.4.** *Let  $g \in \mathbf{BV}[-1,1]$  be an even function, and let  $f := \mathcal{F}(PV - g(s)/s)$ . Let  $X$  be a UMD space, and let  $-iA$  be the generator of a  $C_0$ -group  $U = (U(s))_{s \in \mathbb{R}}$  on  $X$ . Then  $f(A)$  is bounded, and*

$$f(A)x = PV - \int_{-1}^1 g(s)U(s)x \frac{ds}{s} := \lim_{\varepsilon \searrow 0} \int_{\varepsilon \leq |t| \leq 1} g(s)U(s)x \frac{ds}{s} \quad (x \in X)$$

*Proof.* Consider the function  $g_\varepsilon = g(\mathbf{1} - \mathbf{1}_{(-\varepsilon, \varepsilon)})$ . Then  $g_\varepsilon(1) = g(1)$  and  $\sup_\varepsilon \text{Var}_{[0,1]}(g_\varepsilon) < \infty$ . Let  $f_\varepsilon := \mathcal{F}(g_\varepsilon(s)/s)$ . Clearly  $f_\varepsilon \rightarrow f$  pointwise, and  $\sup_\varepsilon \|f_\varepsilon\|_{H_1^\infty(St_\theta)} < \infty$ , by Lemma 4.3. So the statement follows by Theorem 3.6 and the Convergence Lemma.  $\square$

The last class of examples involve functions that are bounded and holomorphic not only on a strip but on a region

$$V_{\varphi, \theta} := St_\theta \cup \Sigma_\varphi$$

for some  $\theta > 0, \varphi \in (0, \pi/2)$ . (Recall that  $\Sigma_\varphi$  is a double sector and  $St_\theta$  is a horizontal strip. The set  $V_{\varphi, \theta}$  is sometimes called a *Venturi region* — inspired by the Venturi tube from fluid dynamics. We shall need the following fact.

**Lemma 4.5.** *Let  $0 < \theta < \varphi < \pi/2$ . Then there is a constant  $c = c(\theta, \varphi)$  such that*

$$\sup\{|zf'(z)| \mid z \in \Sigma_\theta\} \leq c \|f\|_{H^\infty(\Sigma_\varphi)} \quad (f \in H^\infty(\Sigma_\varphi)).$$

*Analogously, for any  $0 < \theta < \varphi < \pi$  there is a constant  $c = c(\theta, \varphi)$  such that*

$$\sup\{|zf'(z)| \mid z \in S_\theta\} \leq c \|f\|_{H^\infty(S_\varphi)} \quad (f \in H^\infty(S_\varphi)).$$

*Proof.* The proof uses the Cauchy integral integral formula, and is easy. See [27, Section 4] or [25, Lemma 8.2.6].  $\square$

Now we can state the theorem, apparently an analogue for unbounded groups of the Hieber–Prüss Theorem 1.2.

**Theorem 4.6.** *Let  $X$  be a UMD space, and let  $-iA$  be the generator of a strongly continuous group  $U = (U(s))_{s \in \mathbb{R}}$  on  $X$ . Let  $\theta > \theta(U)$  and  $\varphi \in (0, \pi/2)$ . Then  $A$  has a bounded  $H^\infty(V_{\varphi, \theta})$ -calculus. More precisely, there is a constant  $c > 0$  such that*

$$\|f(A+r)\| \leq c \|f\|_\infty \quad (r \in \mathbb{R}, f \in H^\infty(V_{\varphi, \theta}))$$

where  $\|f\|_\infty$  denotes the supremum norm of  $f$  on  $V_{\varphi, \theta}$ .

*Proof.* By virtue of Theorem 3.6 it suffices to show the continuous inclusion

$$H^\infty(V_{\varphi, \theta}) \subset H_1^\infty(St_{\theta'})$$

for  $\theta' \in (\theta(U), \theta)$ . But this follows easily from Lemma 4.5, a).  $\square$

## 5. SECTORIAL OPERATORS AND COSINE GENERATORS

We briefly discuss applications of the previous results to sectorial operators and operators that generate cosine functions.

An operator  $A$  on a Banach space  $X$  is called *sectorial* of angle  $\omega \in [0, \pi)$  if

$$\{z \in \mathbb{C} \setminus \{0\} \mid \omega < |\arg z| \leq \pi\} \subset \varrho(A)$$

and for every  $\omega' \in (\omega, \pi)$

$$M(A, \omega') := \sup\{\|zR(z, A)\| \mid \omega' \leq |\arg z| \leq \pi\} < \infty.$$

The minimum of all  $\omega$  such that  $A$  is sectorial of angle  $\omega$  is denoted by  $\omega_{sect}(A)$ , and is called the *sectoriality angle*. Basic properties of sectorial operators can be found in [29, Chapter 1] or [25, Chapter 2]. As in the case of (strong) strip-type operators one has a certain functional calculus for sectorial operators, a detailed description of which is in [25]. If the sectorial operator  $A$  is injective, then  $\log A$  is defined as

well as  $f(A)$  for each  $f \in H^\infty(S_\varphi)$ , where  $\varphi \in (\omega, \pi)$ . Moreover,  $\log(A)$  is a strong strip-type operator, with  $\omega_{sst}(\log(A)) = \omega_{sect}(A)$ , and there is a composition rule:

$$f(\log(A)) = (f \circ \log z)(A) \quad (\omega \in (\omega_A, \pi), f \in H^\infty(St_\omega)) \quad (3)$$

See [25, Chapter 4] for these results. The sectorial operator  $A$  is said to have *bounded imaginary powers* if it is injective and  $-i \log(A)$  generates a  $C_0$ -group  $U$ . In this case  $U(s) = A^{-is}$ ,  $s \in \mathbb{R}$ . One writes  $\theta_A := \theta(U)$  for the type of this group. By a result of PRÜSS and SOHR one has  $\omega_{sect}(A) \leq \theta_A$ , see [25, Corollary 4.3.4] or [22] for an alternative proof. In the case that  $X = H$  is a Hilbert space then  $\theta_A = \omega_{sect}(A)$  (a result by MCINTOSH, see also [25, Corollary 4.3.5]). If  $A$  has bounded imaginary powers one writes  $A \in \text{BIP}(X)$ .

Let  $\varphi \in (0, \pi)$  and let

$$H_{\log}^\infty(S_\varphi) := \{f \in H^\infty(S_\varphi) \mid |z(\log z)f'(z)| \in H^\infty(S_\varphi)\}$$

with the obvious norm. Then we have the following theorem.

**Theorem 5.1.** *Let  $X$  be a UMD space and let  $A \in \text{BIP}(X)$  such that  $\theta_A < \pi$ . Then the following assertions hold.*

- a)  *$A$  has a bounded  $H_{\log}^\infty(St_\varphi)$ -calculus, for every  $\varphi \in (\theta_A, \pi)$ .*
- b) *If  $X = H$  is a Hilbert space, then  $A$  has a bounded  $H^\infty(St_\varphi)$ -calculus, for every  $\varphi \in (\omega_A, \pi)$ .*

Statement b) is due to MCINTOSH [30], statement a) is new. Note that if  $f \in H^\infty(St_\varphi)$  one anyway has that  $zf'(z)$  is bounded on each smaller sector (see the proof of Theorem 4.6 above).

*Proof.* By virtue of the composition rule (3) b) follows from Corollary 3.7 and a) follows from Theorem 3.6. Note that the mapping

$$(f \longmapsto f \circ (\log z)) : H_1^\infty(St_\varphi) \longrightarrow H_{\log}^\infty(S_\varphi)$$

is an isometric isomorphism. □

Let us turn to a different application. Note that if  $-A$  generates a bounded  $C_0$ -semigroup, then  $A$  is sectorial of angle  $\leq \pi/2$ .

**Theorem 5.2.** *Let  $-A$  generate an exponentially stable semigroup  $T$  on a UMD space  $X$ . If  $T$  is a group, then for every  $\varphi \in (\pi/2, \pi)$  the  $H^\infty(St_\varphi)$ -calculus for  $A$  is bounded.*

*Proof.* This follows from Theorem 4.6 by rotating and shifting. Furthermore, one needs a certain compatibility of functional calculi (for sectorial operators and for (rotated, shifted) strip-type operators). This compatibility is straightforward on the level of elementary calculi by path deforming, and so holds for extended calculi, cp. [25, Proposition 1.2.7]. □

Our techniques allow new proofs of the theorems of MONNIAUX [31] and DORE and VENNI [20]; this is discussed at length in [24].

Let us turn to generators of cosine functions. We shall be sketchy in providing the background, referring to [1, Section 3.14-3.16] for the general facts, and to [26] for functional calculus matters.

A cosine function on a Banach space  $X$  is a strongly continuous mapping  $\text{Cos} : \mathbb{R} \rightarrow \mathcal{L}(X)$  that satisfies the identity

$$\text{Cos}(t+s) + \text{Cos}(t-s) = 2 \text{Cos}(t) \text{Cos}(s) \quad (t, s \in \mathbb{R})$$

as well as  $\text{Cos}(0) = I$ . One can prove from this that a cosine function is exponentially bounded, so

$$\theta(\text{Cos}) := \inf \{ \omega \geq 0 \mid \exists M \geq 1 : \|\text{Cos}(t)\| \leq M e^{\omega|t|}, t \in \mathbb{R} \} < \infty.$$

The *generator* of a cosine function  $\text{Cos}$  is defined as the unique operator  $A$  such that

$$\lambda R(\lambda^2, A) = \int_0^\infty e^{-\lambda t} \text{Cos}(t) dt \quad (\lambda > \theta(\text{Cos})). \quad (4)$$

The cosine function then provides solutions to the second-order abstract Cauchy problem

$$u''(t) = Au, \quad u(0) = x, \quad u'(0) = 0.$$

From (4) it follows that  $B := -A$  is an operator of *parabola type*  $\omega_0 := \theta(\text{Cos})$ , by which we mean that  $\sigma(B) \subset \overline{\Pi_{\omega_0}}$  and for every  $\omega > \omega_0$  there exists  $M_\omega$  such that

$$\|R(\mu, A)\| \leq \frac{M}{\sqrt{|\mu|} (|\text{Im} \sqrt{\mu}| - \omega)} \quad (\mu \notin \overline{\Pi_\omega}).$$

(Here  $\sqrt{\mu}$  denotes any choice of a square root of  $\mu$ .) As for strong strip-type or sectorial operators, there is a natural holomorphic functional calculus associated with such parabola-type operators. The procedure is canonical: one considers holomorphic functions  $f$  living on parabolas  $\Pi_\omega$  with  $\omega > \omega_0$ . If such a function has good decay at infinity, one may define

$$f(B) := \frac{1}{2\pi i} \int_{\partial \Pi_{\omega'}} f(z) R(z, B) dz$$

where  $\omega' \in (\omega_0, \omega)$  is arbitrary. This gives a primary calculus, and by regularisation [25, Section 1.2] one extends this to a large algebra of meromorphic functions on  $\Pi_\omega$ , including in particular  $H^\infty(\Pi_\omega)$ . If  $A = -B$  happens to generate a cosine function, one has

$$\text{Cos}(t) = \cos(t\sqrt{z})(B) \quad (t \in \mathbb{R}).$$

Note that since  $\cos$  is even,  $\cos(t\sqrt{z})$  is a well-defined bounded holomorphic function on  $\Pi_\omega$ . See [26] for proofs and more information.

The idea of reducing the second-order equation to a first-order system leads to the notion of *phase space*. Namely, the operator matrix

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

is the generator of a  $C_0$ -group  $\mathcal{U}$  on a space of the form  $\mathcal{X} := X \times V \subset X \times X$ , the so-called phase space. KISYNSKI has shown that there is a unique subspace of  $X \times X$  with this property, and therefore it was proposed recently that the space  $V$  (which apparently determines the phase space) should be called the *Kisyński space*. It was observed in [24, Appendix] that  $\theta(\mathcal{U}) = \theta(\text{Cos})$  and that  $V$  and hence  $\mathcal{X}$  is UMD (Hilbert) if  $X$  is UMD (Hilbert). (This was known before, but somehow by an a posteriori argument. See [24] for details on this admittedly cryptic remark.) As obviously

$$\mathcal{A}^2 = \begin{pmatrix} A & 0 \\ 0 & A|_V \end{pmatrix} \quad \text{on} \quad \mathcal{D}(A) \times \mathcal{D}(A|_V)$$

is in diagonal form, properties of  $A$  can be deduced from properties of  $\mathcal{A}^2$ . In particular, one will get

$$\begin{pmatrix} f(-A) & 0 \\ 0 & f(-A)|_V \end{pmatrix} = f(-\mathcal{A}^2) = f(z^2)(i\mathcal{A})$$

for sensible holomorphic functions  $f$  on the parabola  $\Pi_\omega$ ,  $\omega > \theta(\text{Cos}) = \theta(\mathcal{U})$ . Writing  $\mathcal{B} := i\mathcal{A}$  we have that  $-i\mathcal{B}$  generates  $\mathcal{U}$  and we can apply our results from above. As in the strip case we write

$$H_1^\infty(\Pi_\omega) := \{f \in H^\infty(\Pi_\omega) \mid zf'(z) \in H^\infty(\Pi_\omega)\},$$

and endow it with the canonical norm.

**Theorem 5.3.** *Let  $A = -B$  generate a cosine function  $\text{Cos}$  on the UMD space  $X$ , and let  $\omega > \theta(\text{Cos})$ .*

a) *The operator  $B$  has a bounded  $H_1^\infty(\Pi_\omega)$ -calculus.*

b) *If  $X = H$  is a Hilbert space, then  $B$  has a bounded  $H^\infty(\Pi_\omega)$ -calculus.*

*The same statements are true for the operator  $B|_V$ .*

*Proof.* By our remarks above,  $f(B) \oplus f(B|_V) = f(z^2)(\mathcal{B})$ . Now, writing  $w = z^2$  we see that

$$2wf'(w) = z \frac{d}{dz} (f(z^2))$$

so  $f \in H_1^\infty(\Pi_\omega)$  if and only if  $f(z^2) \in H_1^\infty(\text{St}_\omega)$ . Since  $-i\mathcal{B}$  generates the group  $\mathcal{U}$  on the UMD space  $\mathcal{X} = X \times V$ , we can apply Theorem 3.6, a) to conclude that  $f(z^2)(\mathcal{B})$  is a bounded operator for all  $f \in H_1^\infty(\Pi_\omega)$ , and this proves a).

If  $X = H$  is a Hilbert space, then  $V$  is also a Hilbert space, and we may apply Corollary 3.7 to prove claim b).  $\square$

**Remark 5.4.** Part b) of Theorem 5.3 improves the known results [25, Section 7.4] in that we now have the additional information that the group on  $V \times H$  has the same growth type as the original cosine function, and so no shifting is needed any more.

If  $B$  is an operator of parabola-type  $\omega_0$ , then for large  $\lambda > 0$  the operator  $\lambda + B$  will be sectorial. In fact, simple geometry yields that for  $\theta \in (0, \pi/2]$  and  $\omega \geq 0$

$$(\omega/\sin\theta)^2 + \Pi_\omega \subset S_\theta, \quad (5)$$

and some further computation shows that the operator  $B_\theta := B + (\omega_0/\sin\theta)^2$  is sectorial of angle  $\theta$ , see [26, Proposition 7.6]. Furthermore, if  $-B$  generates a cosine function then so does  $-B_\theta$ , by perturbation theory [1, Corollary 3.14.10].

**Theorem 5.5.** *Let  $A = -B$  be the generator of a cosine function  $\text{Cos}$  on the UMD space  $X$ . Let  $\theta \in (0, \pi/2]$  and set  $B_\theta := -A + (\omega_0/\sin\theta)^2$ , where  $\omega_0 := \theta(\text{Cos})$  is the exponential growth type of the cosine function. Then for every  $\varphi \in (\theta, \pi)$  the operator  $B_\theta$  has bounded  $H^\infty(S_\varphi)$ -calculus.*

*Proof.* Choose  $\theta' \in (\theta, \varphi)$  and define  $\omega := \omega_0 \sin\theta'/\sin\theta > \omega_0$  and  $\lambda := (\omega/\sin\theta')^2 = (\omega_0/\sin\theta)^2$ . Let  $f \in H^\infty(S_\varphi)$  and define  $g(z) := f(\lambda + z)$ . Then obviously  $f(B_\theta) = f((\omega_0/\sin\theta)^2 + B) = f(\lambda + B) = g(B)$ . Now  $g \in H_1^\infty(\Pi_\omega)$ . To see this note first that by (5)

$$\lambda + \Pi_\omega = (\omega/\sin\theta')^2 + \Pi_\omega \subset S_{\theta'} \subset S_\varphi$$

so  $g$  is bounded on  $\Pi_\omega$  by  $\|f\|_{H^\infty(S_\varphi)}$ . Moreover,

$$|zg'(z)| = \left| \frac{z}{z+\lambda} \right| |(z+\lambda)f'(z+\lambda)| \leq c \left| \frac{z}{z+\lambda} \right| \|f\|_{H^\infty(S_\varphi)} \quad (z \in \Pi_\omega)$$

for some constant  $c$ , by Lemma 4.5; since  $-\lambda \notin \overline{\Pi_\omega}$ , the first factor is bounded on  $\Pi_\omega$ . We may now apply Theorem 5.3 to conclude that  $f(B_\theta) = g(B)$  is bounded.  $\square$

**Corollary 5.6.** *Let  $A$  be the generator of a cosine function on the UMD space  $X$ . If  $B := -A$  is sectorial and invertible, then  $B$  has bounded  $H^\infty(S_\varphi)$ -calculus for every  $\varphi \in (\omega_{sect}(B), \pi)$ .*

*Proof.* Fix  $\varphi \in (\omega_{sect}(B), \pi)$ . Since  $B$  is assumed to be invertible, standard perturbation theory [25, Corollary 5.5.5] shows that it suffices to prove that  $\lambda + B$  has bounded  $H^\infty(S_\varphi)$ -calculus, for some  $\lambda > 0$ . By Theorem 5.3,  $\lambda := (\omega_0/\sin\theta)^2$  will do, where  $\omega_0$  is the exponential growth type of the cosine function generated by  $A = -B$  and  $\theta \in (0, \varphi)$  is arbitrary.  $\square$

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