

MATH 3375/5375: Hydrodynamic Stability Vector Calculus Revision

Vector Notation (in Cartesian coordinates)

Gradient of a scalar function:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right).$$

Divergence of a vector function:

$$\nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z},$$

where vector $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$.

Curl of a vector function:

$$\nabla \times \mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\mathbf{k}.$$

[Note: when written by hand, as in lectures, the curl of \mathbf{a} is often written as $\nabla \wedge \mathbf{a}$.]

The Laplacian operator on a scalar function ϕ is written as $\nabla^2\phi$ and defined as

$$\nabla^2\phi = \nabla \cdot (\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

Vector Identities

- (i) $\nabla(\phi\mathbf{a}) = \phi\nabla \cdot \mathbf{a} + \nabla\phi \cdot \mathbf{a}$
- (ii) $\nabla \times (\phi\mathbf{a}) = \nabla\phi \times \mathbf{a} + \phi(\nabla \times \mathbf{a})$
- (iii) $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$
- (iv) $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b})$
- (v) $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$
- (vi) $\nabla \times \nabla\phi = 0$ for any scalar function ϕ
- (vii) $\nabla \cdot (\nabla \times \mathbf{a}) = 0$ for any vector \mathbf{a}

Integral Theorems

Divergence Theorem

The divergence theorem relates volume and surface integrals. Suppose a volume V is bounded by a closed surface S . Then if \mathbf{a} is a vector function of position with continuous derivatives:

$$\int \int \int_V \nabla \cdot \mathbf{a} dV = \int \int_S \mathbf{a} \cdot \mathbf{n} dS = \int \int_S \mathbf{a} \cdot d\mathbf{S},$$

where \mathbf{n} is the outward pointing unit normal to S .

Stokes' Theorem

Stokes' theorem relates surface and line integrals. Let C be any simple closed curve in 3d space and let S be *any* surface bounded by C . Then if \mathbf{a} has continuous derivatives:

$$\int \int_S (\nabla \times \mathbf{a}) \cdot \mathbf{n} dS = \int \int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} = \oint_C \mathbf{a} \cdot d\mathbf{s},$$

where C is traversed in the positive direction.

Green's Theorem in the Plane

This is a special case of Stokes' theorem, restricted to the plane. Over a region S in the plane with boundary C , Green's theorem states that if M and N are continuous functions of x and y having continuous derivatives:

$$\oint_C M(x, y)dx + N(x, y)dy = \int \int_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

Scalar and Vector Potentials

If a vector field \mathbf{a} has zero curl (i.e. $\nabla \times \mathbf{a} = 0$) then \mathbf{a} can be expressed as the gradient of a scalar function, known as the scalar potential; i.e. there exists a scalar function ϕ such that $\mathbf{a} = \nabla\phi$. This is the converse of vector identity (vi).

If a vector field \mathbf{b} has zero divergence (i.e. $\nabla \cdot \mathbf{b} = 0$) then \mathbf{b} can be expressed as the curl of a vector function, known as the vector potential; i.e. there exists a vector function \mathbf{a} such that $\mathbf{b} = \nabla \times \mathbf{a}$. This is the converse of vector identity (vii).

Other Coordinate Systems

Cylindrical Polar Coordinates (r, θ, z)

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{\partial f}{\partial z} \mathbf{e}_z$$

$$\nabla \cdot \mathbf{a} = \frac{1}{r} \frac{\partial(r a_r)}{\partial r} + \frac{1}{r} \frac{\partial a_\theta}{\partial \theta} + \frac{\partial a_z}{\partial z}$$

$$\nabla \times \mathbf{a} = \frac{1}{r} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ a_r & r a_\theta & a_z \end{vmatrix} = \left(\frac{1}{r} \frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z} \right) \mathbf{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \right) \mathbf{e}_\theta + \frac{1}{r} \left(\frac{\partial(r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta} \right) \mathbf{e}_z$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Polar Coordinates (r, θ, ϕ)

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi$$

$$\nabla \cdot \mathbf{a} = \frac{1}{r^2} \frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta a_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial a_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{a} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\theta & r \sin \theta \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ a_r & r a_\theta & r \sin \theta a_\phi \end{vmatrix}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$