Embedded asymptotic classes

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Outline

Introduction

- An asymptotic class is a class of finite structures such that the sizes of the definable sets are uniformly "controlled" (as a proportion of the size of the ambient structure).
- I'll be talking about one of several variations of the notion of a 1-dimensional asymptotic class - that of an *embedded asymptotic class*.
- This is a work in progress, joint with Dugald Macpherson.

The motivating theorem

Theorem (Chatzidakis, van den Dries, Macintyre)

Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be a formula in the language of rings.

(i) *There is a positive constant C and a finite set D of pairs (d, μ) with $d \in \{0, 1, \ldots, n\}$ and $\mu \in \mathbb{Q}^{>0}$ such that for each finite field $\mathbb{F}_q$, where q is a prime power, and each $\bar{a} \in \mathbb{F}_q^m$, if the set $\varphi(\mathbb{F}_q^n, \bar{a})$ is nonempty, then for some $(d, \mu) \in D$*

$$\left| \left| \varphi\left(\mathbb{F}_q^n, \bar{a}\right) \right| - \mu q^d \right| < C q^{d - \frac{1}{2}}.$$

(ii) *For every $(d, \mu) \in D$, there is a formula $\psi_{d,\mu}(y_1, \ldots, y_m)$ in the language of rings such that for each q, $\psi_{d,\mu}\left(\mathbb{F}_q^m\right)$ consists of those $\bar{a} \in \mathbb{F}_q^m$ for which the corresponding inequality holds.*

1-dimensional asymptotic classes

Definition (MacPherson and Steinhorn)

Let $\mathscr{L}$ be a first-order language, and $\mathscr{C}$ a collection of finite $\mathscr{L}$-structures. $\mathscr{C}$ is a *1-dimensional asymptotic class* if for every $\mathscr{L}$-formula $\varphi(\overline{x}, \overline{y}) := \varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$:

(i) There is a positive constant C and a finite set D of pairs (d, μ) with $d \in \{0, \ldots, n\}$ and $\mu \in \mathbb{R}^{>0}$, such that for every $M \in \mathscr{C}$ and $\overline{a} \in M^m$, if $\varphi(M^n, \overline{a})$ is non-empty, then for some $(d, \mu) \in D$ we have

$$\Big| |\varphi(M^n, \overline{a})| - \mu |M|^d \Big| \leq C |M|^{d-\frac{1}{2}}.$$

(ii) For every $(d, \mu) \in D$, there is an $\mathscr{L}$-formula $\varphi_{d,\mu}(\overline{y})$ such that for all $M \in \mathscr{C}$, $\varphi_{d,\mu}(M^m)$ consists of those $\overline{a} \in M^m$ for which the corresponding inequality holds.

Some notation

- Let $\mathscr{L}$ be a first order language, and $\mathscr{C}$ a collection of finite $\mathscr{L}$-structures.
- $(\mathscr{C}, \bar{y}) := \left\{(M, \bar{a}) | M \in \mathscr{C}, \bar{a} \in {}^{\bar{y}}M\right\}$.
- A partition $\mathscr{S}$ of $(\mathscr{C}, \bar{y})$ is $\emptyset$-*definable* if for each $S \in \mathscr{S}$ there exists an $\mathscr{L}$-formula $\varphi_S(\bar{y})$ without parameters such that

$$\varphi_S(M) = \left\{\bar{b} \in M^{|\bar{y}|} | \left(M, \bar{b}\right) \in S\right\}$$

for each $M \in \mathscr{C}$.

Multidimensional asymptotic classes

Definition (Ascombe, MacPherson, Steinhorn, Wolf)

Let $\mathscr{C}$ be a class of finite $\mathscr{L}$-structures, and R any set of functions $\mathscr{C} \longrightarrow \mathbb{R}_{\geq 0}$. $\mathscr{C}$ is an *R-multidimensional asymptotic class* (R-*MAC*) if for every formula $\varphi(\overline{x}, \overline{y})$ there is a finite $\emptyset$-definable partition $\mathscr{S}$ of $(\mathscr{C}, \overline{y})$ and an indexed set $H_{\mathscr{S}} := \{h_S \in R | S \in \mathscr{S}\}$ such that

$$\left| \left| \varphi\left(M^{|\overline{x}|}; \overline{b}\right) \right| - h_S(M) \right| = o(h_S(M))$$

for $\left(M, \overline{b}\right) \in S$ as $|M| \longrightarrow \infty$. The functions h_S are called the *measuring functions* of $\varphi(\overline{x}; \overline{y})$. When R is understood, we just say that $\mathscr{C}$ is a *MAC*.

Some more notation

- Let $\mathscr{L}$ be a first-order language with signature σ, P a predicate not in σ, and $\mathscr{L}^+$ the language with signature $\sigma \cup \{P\}$.
- If M is an $\mathscr{L}$-structure and A is a (finite) substructure of M, we may view (M, A) as an $\mathscr{L}^+$-structure (i.e. with $P^M = A$).
- If $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ is an $\mathscr{L}^+$ formula and $\bar{a} \in M^m$, we denote by $\varphi^M(A^n, \bar{a})$ the set

$$\left\{\bar{b} \in A^n : (M, A) \models \varphi\left(\bar{b}, \bar{a}\right)\right\} = \left\{\bar{b} : (M, A) \models P\left(\bar{b}\right) \wedge \varphi\left(\bar{b}, \bar{a}\right)\right\}.$$

Definable partitions, renewed

- If M is an $\mathscr{L}$-structure and $\mathscr{C}_0$ is a class of finite substructures A of M, let
$$\mathscr{C} := \{(M, A) : A \in \mathscr{C}_0\}$$
be the corresponding class of $\mathscr{L}^+$-structures.
- As above, for any tuple $\bar{y}$ we define
$$(\mathscr{C}, \bar{y}) := \{(M, A, \bar{a}) : (M, A) \in \mathscr{C}, \bar{a} \in M^{|\bar{y}|}\}.$$
- We say that a partition $\mathscr{S}$ of $(\mathscr{C}, \bar{y})$ is $\emptyset$-*definable* if for each $S \in \mathscr{S}$ there is an $\mathscr{L}^+$-formula $\varphi_S(\bar{y})$ (with no parameters) such that
$$\varphi_S(M^{|\bar{y}|}) = \{\bar{b} \in M^{|\bar{y}|} : (M, A, \bar{b}) \in S\},$$
for each $(M, A) \in \mathscr{C}$.

Embedded MACs

Definition

Let M be an $\mathscr{L}$-structure, and $\mathscr{C}_0$ a collection of finite substructures A of M. View each (M, A) as an $\mathscr{L}^+$-structure, and let $\mathscr{C} = \{(M, A) : A \in \mathscr{C}_0\}$. Let R be any set of functions $\mathscr{C} \to \mathbb{R}_{\geq 0}$. $\mathscr{C}$ is an *embedded R-MAC* if for every $\mathscr{L}^+$-formula $\varphi(\overline{x}; \overline{y})$ there is a finite $\emptyset$-definable partition $\mathscr{S}$ of $(\mathscr{C}, \overline{y})$ and an indexed set $H_{\mathscr{S}} := \{h_S \in R : S \in \mathscr{S}\}$ such that

$$\left| \left| \varphi^M \left(A^{|\overline{x}|}; \overline{b} \right) \right| - h_S(A) \right| = o(h_S(A))$$

for $\left(M, A, \overline{b}\right) \in S$ as $|A| \to \infty$.

Remark

Let $\mathscr{C}$ and $\mathscr{C}_0$ be as in the previous definition. If $\mathscr{C}$ is an embedded R-MAC, then $\mathscr{C}_0$ is at least a weak R-MAC, meaning that the partition $\mathscr{S}$ exists but may not be $\emptyset$-definable. More assumptions are needed to guarantee that $\mathscr{C}_0$ is, in fact, an R-MAC.

Full embeddedness

Let N be a $\emptyset$-definable $\mathscr{L}$-substructure of M, an $\mathscr{L}^+$-substructure ($\mathscr{L} \subset \mathscr{L}^+$).

- N is *stably embedded* in M if for every $\mathscr{L}^+$-formula $\varphi(\overline{x}, \overline{y})$ there are finitely many $\mathscr{L}^+$-formulae $\psi_1(\overline{x}, \overline{z}), \ldots, \psi_k(\overline{x}, \overline{z})$ such that for any $\overline{a} \in M^{|\overline{y}|}$ there are i and $\overline{b} \in N^{|\overline{z}|}$ for which

$$M \vDash \forall \overline{x} \in N(\varphi(\overline{x}, \overline{a}) \longleftrightarrow \psi_i(\overline{x}, \overline{b})).$$

- N is *canonically embedded* in M if for every $\mathscr{L}^+$-formula $\varphi(\overline{x})$ there is an $\mathscr{L}$-formula $\psi(\overline{x})$ such that for all $\overline{a} \in N^{|\overline{x}|}$,

$$M \vDash \varphi(\overline{a}) \Leftrightarrow N \vDash \psi(\overline{a}).$$

- N is *fully embedded* in M if it is both stably and canonically embedded in M.

"Transfer" lemma

Lemma

Let $\mathscr{L}$, P, and $\mathscr{L}^+$ be as in the definition of an embedded R-MAC. Given an R-MAC $\mathscr{C}_0$ consisting of finite substructures of some $\mathscr{L}$-structure M, consider the class of $\mathscr{L}^+$-structures $\mathscr{C} := \{(M, A) : A \in \mathscr{C}_0\}$. If in every ultraproduct $(M^*, P(M^*))$ of $\mathscr{C}$, $P(M^*)$ is fully embedded in M^*, then $\mathscr{C}$ is an embedded R-MAC.

Sketch of proof

- Fix $\varphi(\overline{x}, \overline{y}) \in \mathscr{L}^+$.
- Find finitely many $\mathscr{L}^+$-formulae $\psi_1(\overline{x}, \overline{z}), \ldots, \psi_k(\overline{x}, \overline{z})$ such that for each $P(M) \in \mathscr{C}_0$,

$$(M, P(M)) \models \forall \overline{y} \exists \overline{z} \in P \left(\bigvee_i \forall \overline{x} \in P(\varphi(\overline{x}, \overline{y}) \longleftrightarrow \psi_i(\overline{x}, \overline{z})) \right).$$

- For each i, find finitely many $\mathscr{L}$-formulae $\rho_{i,j}(\overline{x}, \overline{z})$ such that for each $P(M) \in \mathscr{C}_0$,

$$(M, P(M)) \models \bigvee_j \forall \overline{x}, \overline{z} \in P(\psi_i(\overline{x}, \overline{z}) \longleftrightarrow \rho_{i,j}^P(\overline{x}, \overline{z})).$$

Sketch of proof, continued

- Using coding, obtain an $\mathscr{L}$-formula $\eta(\overline{x};\overline{z},\overline{w})$ such that for all $P(M) \in \mathscr{C}_0$ and $\overline{a} \in M^{|\overline{y}|}$ there are $\overline{b}\overline{c} \in P(M)^{|\overline{z}|+|\overline{w}|}$ such that

$$\varphi^M(P(M),\overline{a}) = \eta(P(M),\overline{b},\overline{c}).$$

- The above allows us to transfer the partition for $\mathscr{C}_0$ to one for $\mathscr{C}$. Writing down the defining formulae given those for the original partition is not difficult.

An example

The class $\mathscr{C} = \{(\widetilde{\mathbb{F}}_{p}, \mathbb{F}_{p^n}) : n \in \mathbb{N}\}$ is an embedded 1-dimensional asymptotic class.

- The class $\mathscr{C}_0 = \{\mathbb{F}_{p^n} : n \in \mathbb{N}\}$ is a 1-dimensional embedded asymptotic class.
- Let $\mathscr{L} = \mathscr{L}_{rings}$, $\mathscr{L}^+ = \mathscr{L} \cup \{P\}$, and $\widehat{\mathscr{L}} = \mathscr{L}^+ \cup \{\sigma\}$ (where σ is a unary function).
- Consider the following "expansion" of $\mathscr{C}$:

$$\widehat{\mathscr{C}} = \left\{ \left(\widetilde{\mathbb{F}}_{p}, \mathbb{F}_{p^n}, x \mapsto x^{p^n} \right) : n \in \mathbb{N} \right\}.$$

- By work of Hrushovski, any ultraproduct $(M^*, P(M^*), \sigma)$ of $\widehat{\mathscr{C}}$ is a model of ACFA_p. By work of Chatzidakis and Hrushovski, $P(M^*)$ is fully embedded in M^*. This is sufficient to apply the lemma.

References

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