An introduction to relational complexity: background, questions, and a few answers

Joshua Wiscons

California State University, Sacramento

Workshop on the Model Theory of Finite and Pseudofinite Structures

Relational Complexity

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Definition (via Homogeneity)

The relational complexity of $\mathbf{X}$ is the least k such that $\mathbf{X}$ is equivalent to a homogeneous structure $(X, S_1, \ldots, S_n)$ with every S_i of arity at most k.

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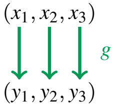

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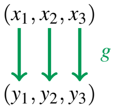

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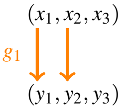

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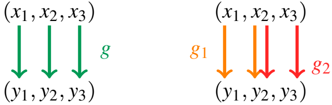

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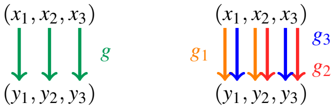

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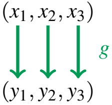

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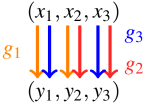

2-equivalence

Note: 3-equivalence implies 2-equivalence

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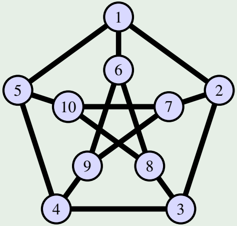
Definition

The relational complexity of a permutation group (X, G) is the smallest k such that for all $n \geq k$, k-equivalence of n-tuples implies equivalence.

A First Example

Example (Petersen Graph)



Let G be the automorphism group of the graph.

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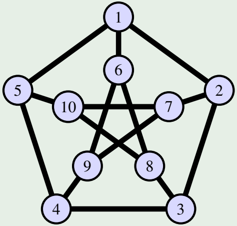

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Question: $(1,2,3) \sim (1,5,4)$?

$$(1,2,3)$$

$$\downarrow\ \downarrow\ \downarrow \quad ?$$

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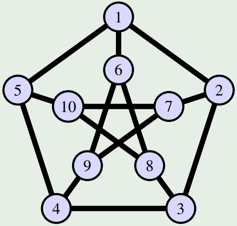

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reflection: $(25)(34)(7\,10)(89)$

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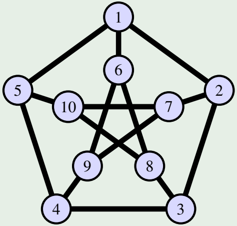

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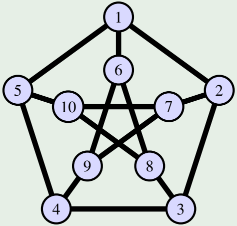

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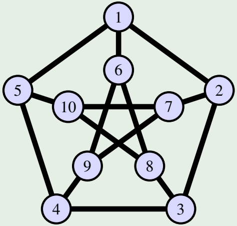

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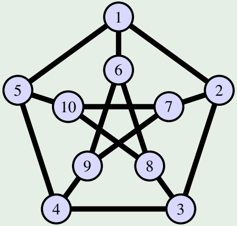

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Question: $(1,3,7) \sim_2 (1,3,9)$?

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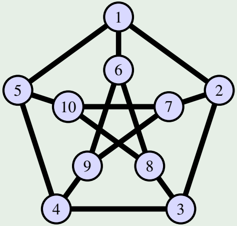

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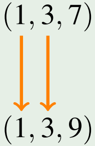

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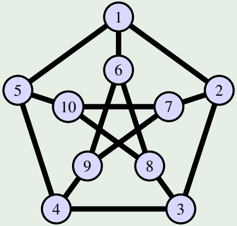

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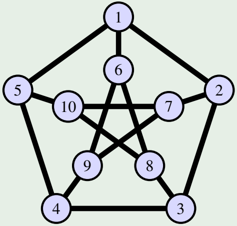

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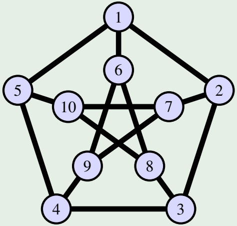

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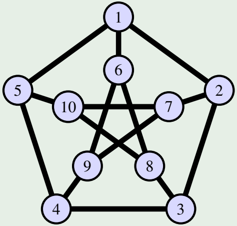

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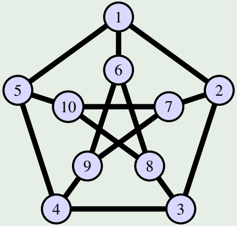

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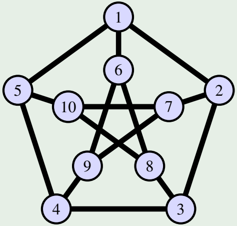

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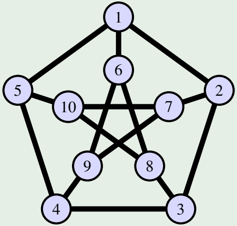

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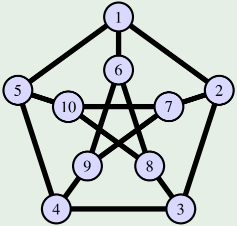

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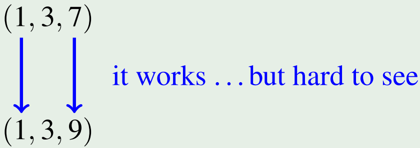

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- Thus, $\mathrm{rc}(G) > 2$.

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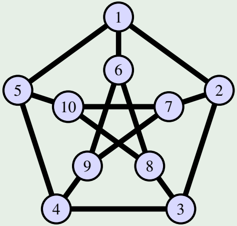

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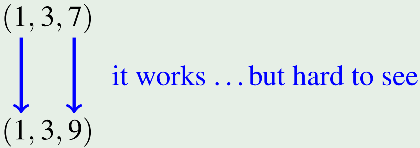
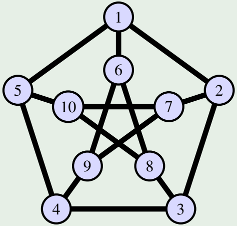


- Thus, 2-equivalence does *not* imply equivalence.
- Thus, $\mathrm{rc}(G) > 2$. It turns out that $\mathrm{rc}(G) = 3$.

A First Example

Example (Petersen Graph)



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$$(1,3,7) \downarrow\downarrow (1,3,9)$$

it works … but hard to see

- Thus, 2-equivalence does *not* imply equivalence.
- Thus, $\mathrm{rc}(G) > 2$. It turns out that $\mathrm{rc}(G) = 3$.

Remark

Be aware that, for example, 2-equivalence may imply 3-equivalence without implying 4-equivalence.

Remarks And More Examples

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Let (X, G) be a *nontrivial* permutation group.

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- Choose a nontrivial $g \in G$.
- Choose distinct $x, y \in X$ with $xg = y$.
- Then (x, x) and (x, y) are 1-equivalent but not 2-equivalent.

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- Thus, $\bar{x}$ and $\bar{y}$ are $|X|$-equivalent.

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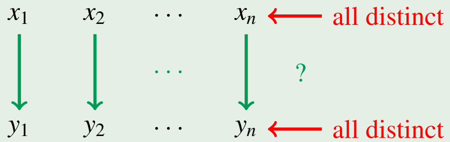

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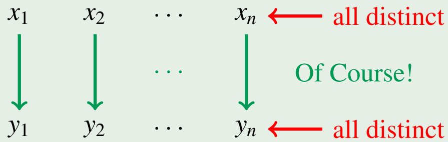

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- Let $\bar{x}, \bar{y}$ be *any* two enumerations of X.
- Key point: $(X, \mathrm{Alt}(n))$ is $(n - 2)$-transitive.
- Thus, $\bar{x}$ and $\bar{y}$ are $(n - 2)$-equivalent.
- So, if $r \leq n - 2$, $\bar{x}$ and $\bar{y}$ are equivalent.

Some Problems (posed by Cherlin)

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   - E.g, what are the finite primitive permutation groups of relational complexity 2?
3. Classify the finite permutation groups of relational complexity at least r when r is "big" compared to $|X|$.

An easy, natural example

Example ($\mathrm{GL}(V)$)

Let $V = \mathbb{F}^d$ with $\mathrm{char}(\mathbb{F}) > 2$.

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   - But, $\bar{x}$ and $\bar{y}$ are *not* equivalent.
2. Upper bound: $\mathrm{rc}(V, \mathrm{GL}(V)) \leq d + 1$

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   - Want to show that $(d + 1)$-equivalence implies equivalence

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Let $V = \mathbb{F}^d$ with $\mathrm{char}(\mathbb{F}) > 2$. Then $\mathrm{rc}(V, \mathrm{GL}(V)) = ?$

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$\mathcal{P}_m(n)$ is the set of all *partitions* of $\{1, \ldots, n\}$ into subsets of (equal) size m.

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$$\sigma\left([x_1 \cdots x_m \mid y_1 \cdots y_m \mid \cdots]\right) = [\sigma(x_1) \cdots \sigma(x_m) \mid \sigma(y_1) \cdots \sigma(y_m) \mid \cdots].$$

A harder, natural example

Problem

Determine the relational complexity of S_n acting on $\mathcal{P}_m(n)$.

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$$\mathcal{P}_3(9): \quad [***\mid ***\mid ***] \implies \mathrm{rc}(S_9) = 5$$
$$\mathcal{P}_2(4): \quad [**\mid **] \implies \mathrm{rc}(S_4) = 2$$
$$\mathcal{P}_3(6): \quad [***\mid ***] \implies \mathrm{rc}(S_6) = 3$$

A harder, natural example

Problem

Determine the relational complexity of S_n acting on $\mathcal{P}_m(n)$.

Some answers are known, but the reasons why aren't so unclear.

$$\mathcal{P}_2(4): \quad [**\,|\,**] \implies \mathrm{rc}(S_4) = 2$$
$$\mathcal{P}_2(6): \quad [**\,|\,**\,|\,**] \implies \mathrm{rc}(S_6) = 3$$
$$\mathcal{P}_2(8): \quad [**\,|\,**\,|\,**\,|\,**] \implies \mathrm{rc}(S_8) = 4$$

$$\mathcal{P}_3(6): \quad [***\,|\,***] \implies \mathrm{rc}(S_6) = 3$$
$$\mathcal{P}_3(9): \quad [***\,|\,***\,|\,***] \implies \mathrm{rc}(S_9) = 5$$

$$\mathcal{P}_2(4): \quad [**\,|\,**] \implies \mathrm{rc}(S_4) = 2$$
$$\mathcal{P}_3(6): \quad [***\,|\,***] \implies \mathrm{rc}(S_6) = 3$$
$$\mathcal{P}_4(8): \quad [****\,|\,****] \implies \mathrm{rc}(S_8) = 5$$

A harder, natural example

Problem

Determine the relational complexity of S_n acting on $\mathcal{P}_m(n)$.

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$$\mathcal{P}_3(6): \quad [***\,|\,***] \implies \mathrm{rc}(S_6) = 3$$
$$\mathcal{P}_3(9): \quad [***\,|\,***\,|\,***] \implies \mathrm{rc}(S_9) = 5$$
$$\mathcal{P}_2(4): \quad [**\,|\,**] \implies \mathrm{rc}(S_4) = 2$$
$$\mathcal{P}_3(6): \quad [***\,|\,***] \implies \mathrm{rc}(S_6) = 3$$
$$\mathcal{P}_4(8): \quad [****\,|\,****] \implies \mathrm{rc}(S_8) = 5$$
$$\mathcal{P}_5(10): \quad [*****\,|\,*****] \implies \mathrm{rc}(S_{10}) = 4$$

A harder, natural example

Problem

Determine the relational complexity of S_n acting on $\mathcal{P}_m(n)$.

Some answers are known, but the reasons why aren't so unclear.

$$\mathcal{P}_2(4): \quad [**\,|\,**] \implies \mathrm{rc}(S_4) = 2$$
$$\mathcal{P}_2(6): \quad [**\,|\,**\,|\,**] \implies \mathrm{rc}(S_6) = 3$$
$$\mathcal{P}_2(8): \quad [**\,|\,**\,|\,**\,|\,**] \implies \mathrm{rc}(S_8) = 4$$
$$\mathcal{P}_3(6): \quad [***\,|\,***] \implies \mathrm{rc}(S_6) = 3$$
$$\mathcal{P}_3(9): \quad [***\,|\,***\,|\,***] \implies \mathrm{rc}(S_9) = 5$$
$$\mathcal{P}_2(4): \quad [**\,|\,**] \implies \mathrm{rc}(S_4) = 2$$
$$\mathcal{P}_3(6): \quad [***\,|\,***] \implies \mathrm{rc}(S_6) = 3$$
$$\mathcal{P}_4(8): \quad [****\,|\,****] \implies \mathrm{rc}(S_8) = 5$$
$$\mathcal{P}_5(10): \quad [*****\,|\,*****] \implies \mathrm{rc}(S_{10}) = 4$$
$$\mathcal{P}_6(12): \quad [******\,|\,******] \implies \mathrm{rc}(S_{12}) \geq 6$$

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

- Set $x = [1234 \mid 5678]$.

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

- Set $x = [1234 \mid 5678]$.
- Every $x' \neq x$ can have one of two relationships with x.

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

- Set $x = [1234 \mid 5678]$.
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$$[1234 \mid 5678]$$

$$[5234 \mid 1678]$$

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

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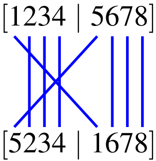

$(1, 3)$-pattern

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

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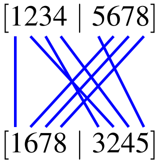

$(1,3)$-pattern

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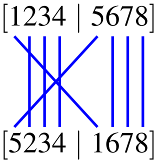


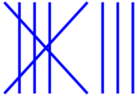
$(1, 3)$-pattern

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

- Set $x = [1234 \mid 5678]$.
- Every $x' \neq x$ can have one of two relationships with x.

$[1234 \mid 5678]$ $\qquad$ $[1234 \mid 5678]$



$[5234 \mid 1678]$ $\qquad$ $[5634 \mid 1278]$

$(1,3)$-pattern

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

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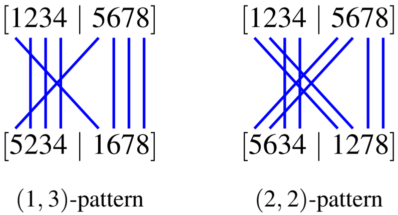

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Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

- Set $x = [1234 \mid 5678]$.
- Every $x' \neq x$ can have one of two relationships with x.
- Let $Y \subset \mathcal{P}_4(8)$ be the partitions that have a $(1,3)$-pattern with x.

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

- Set $x = [1234 \mid 5678]$.
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- Let $Y \subset \mathcal{P}_4(8)$ be the partitions that have a $(1,3)$-pattern with x.
- Every $y \in Y$ is determined by two numbers.

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

- Set $x = [1234 \mid 5678]$.
- Every $x' \neq x$ can have one of two relationships with x.
- Let $Y \subset \mathcal{P}_4(8)$ be the partitions that have a $(1,3)$-pattern with x.
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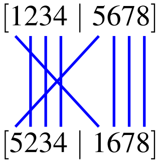

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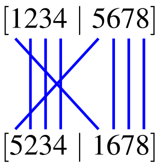

$\rightsquigarrow (1, 5)$

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

- Set $x = [1234 \mid 5678]$.
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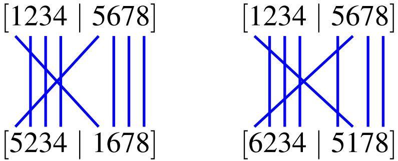
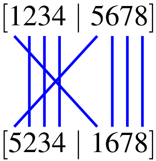


$\rightsquigarrow (1,5)$

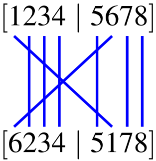
A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

- Set $x = [1234 \mid 5678]$.
- Every $x' \neq x$ can have one of two relationships with x.
- Let $Y \subset \mathcal{P}_4(8)$ be the partitions that have a $(1,3)$-pattern with x.
- Every $y \in Y$ is determined by two numbers.



$\rightsquigarrow (1,5)$



$\rightsquigarrow (1,6)$

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

- Set $x = [1234 \mid 5678]$.
- Every $x' \neq x$ can have one of two relationships with x.
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- Every $y \in Y$ is determined by two numbers.

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

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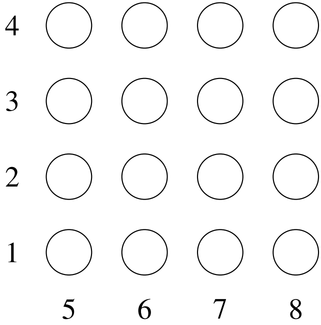

$x = [1234 \mid 5678]$

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

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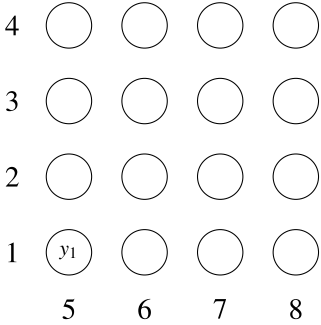

$$x = [1234 \mid 5678]$$
$$y_1 = [5234 \mid 1678]$$

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

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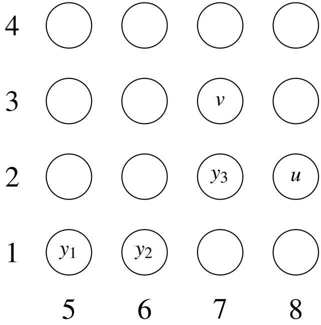

$$\begin{aligned} x &= [1234 \mid 5678] \\ y_1 &= [5234 \mid 1678] \\ y_2 &= [6234 \mid 5178] \\ y_3 &= [1734 \mid 5628] \\ u &= [1834 \mid 5672] \\ v &= [1274 \mid 5638] \end{aligned}$$

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.

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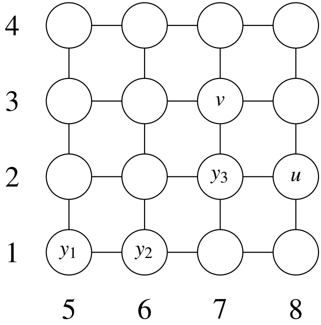

$$x = [1234 \mid 5678]$$
$$y_1 = [5234 \mid 1678]$$
$$y_2 = [6234 \mid 5178]$$
$$y_3 = [1734 \mid 5628]$$
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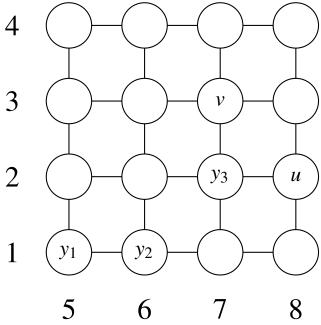
$$v = [1274 \mid 5638]$$

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.



$$
\begin{aligned}
x &= [1234 \mid 5678] \\
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\end{aligned}
$$

A harder, natural (concrete) example

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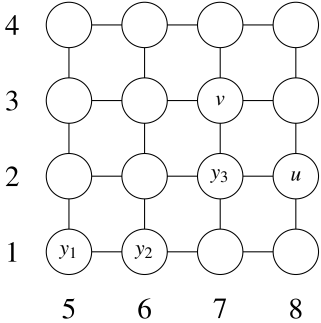

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A harder, natural (concrete) example

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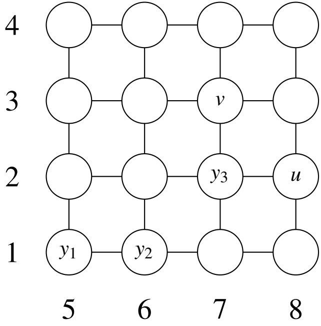


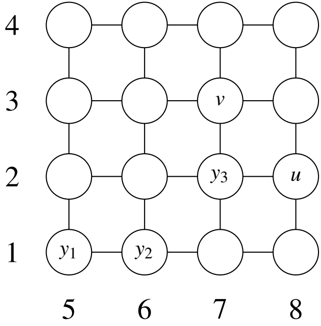
$$x = [1234 \mid 5678]$$
$$y_1 = [5234 \mid 1678]$$
$$y_2 = [6234 \mid 5178]$$
$$y_3 = [1734 \mid 5628]$$
$$u = [1834 \mid 5672]$$
$$v = [1274 \mid 5638]$$

$x \quad y_1 \quad y_2 \quad y_3 \quad u$

$x \quad y_1 \quad y_2 \quad y_3 \quad v$

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.



$$
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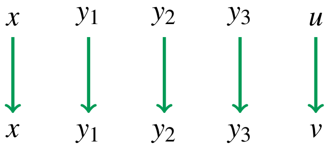
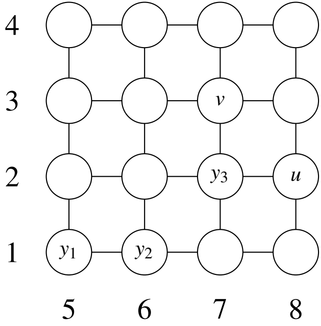
\end{aligned}
$$



Let's look at $\mathcal{P}_4(8)$, i.e. $[****\mid ****]$, and try to see why $\mathrm{rc}(S_8) = 5$.



$$
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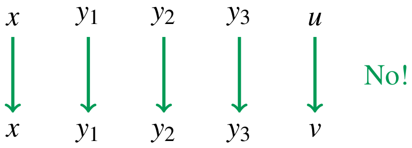
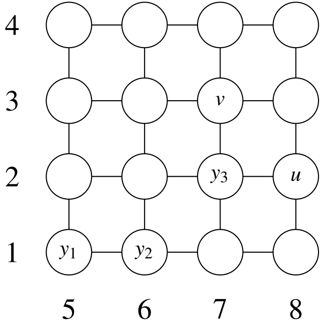


A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.



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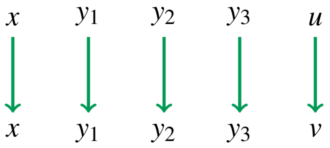
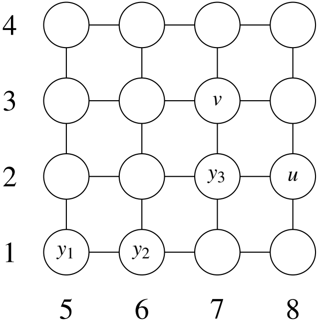


No! The stabilizer of (x, y_1, y_2, y_3) is $\langle (34) \rangle$.

A harder, natural (concrete) example

Let's look at $\mathcal{P}_4(8)$, i.e. $[* * * * \mid * * * *]$, and try to see why $\mathrm{rc}(S_8) = 5$.



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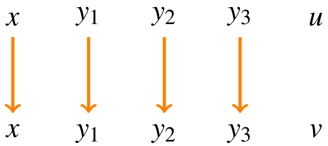

A harder, natural (concrete) example

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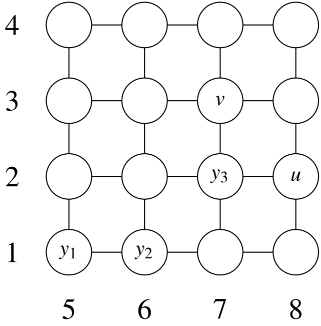

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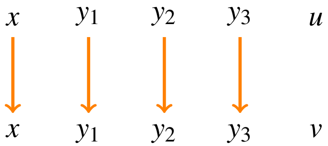

Yes!

A harder, natural (concrete) example

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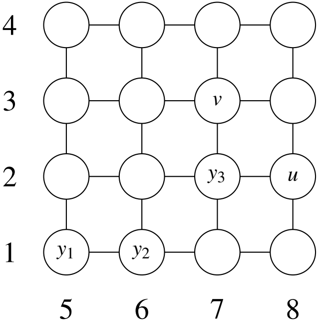

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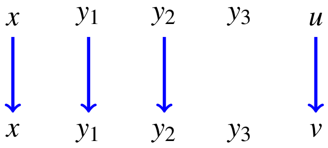

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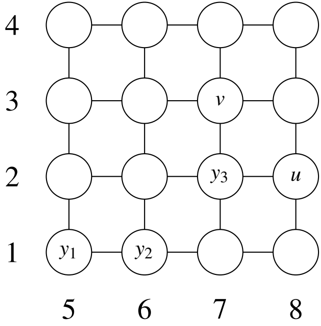

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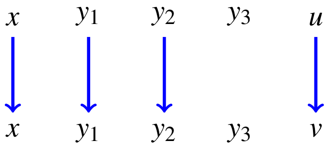

Yes: (23)(78) ("rotation")

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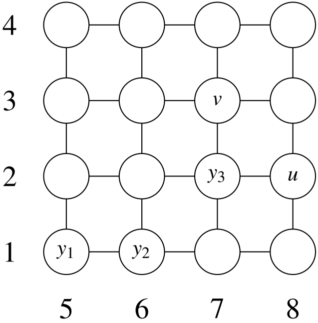

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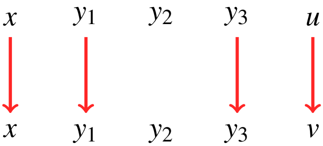

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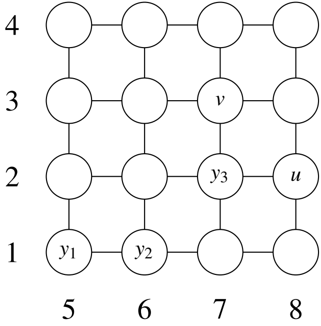

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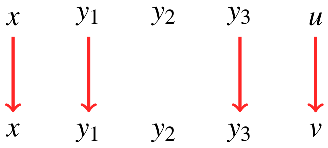

Yes: $(15)(27)(38)(46)$ ("reflection")

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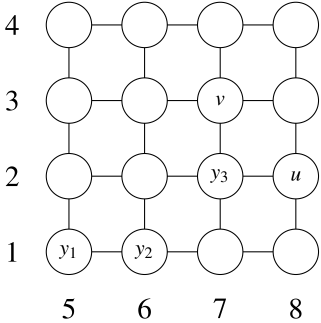

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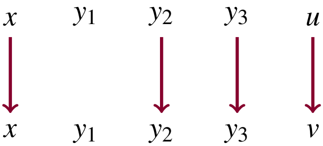

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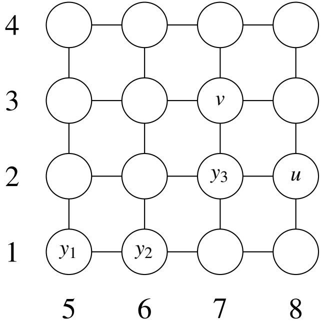

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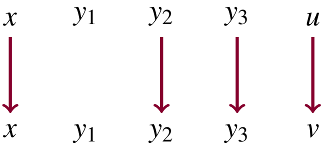

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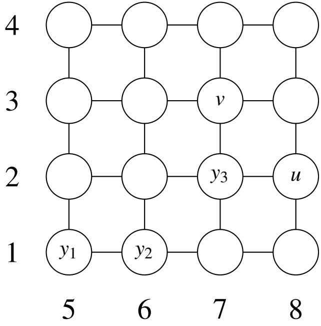

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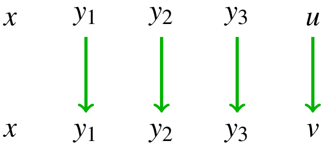

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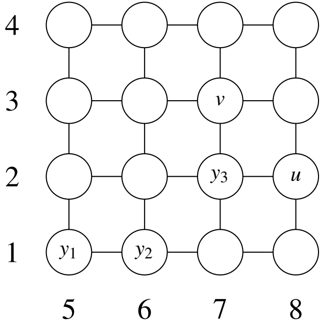

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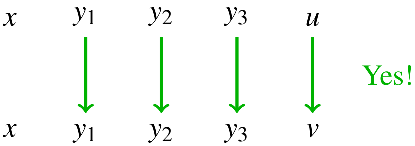

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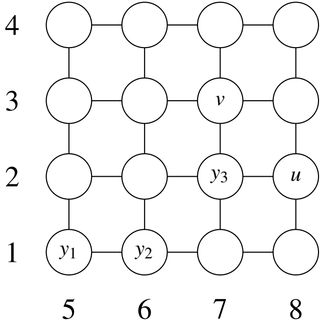

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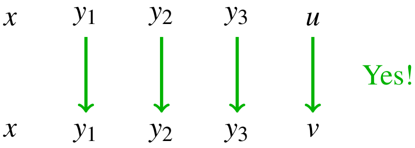

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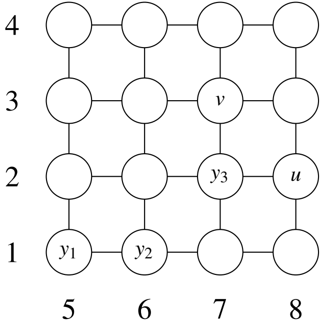

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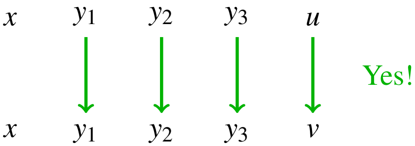

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$$\mathrm{rc}(S_8) \geq 5$$

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W, '16: *The conjecture reduces to the almost simple case.*

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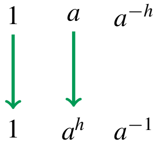

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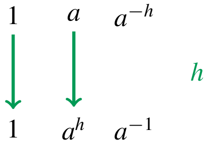

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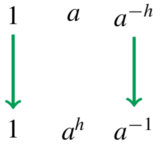

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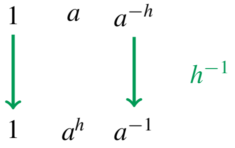

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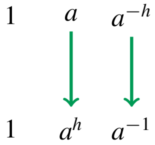

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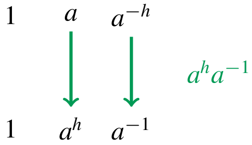

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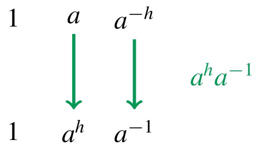

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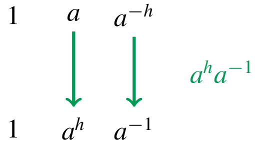

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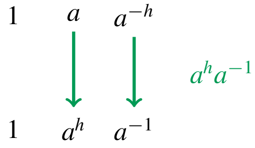

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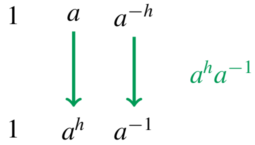

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Punchline: in these cases, complexity equal to 2 can be used to create involutions in a point stabilizer.

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Thank You