Structure and enumeration theorems for hereditary properties in finite relational languages

Caroline Terry

UIC

Model Theory and Pseudofinite Structures Workshop

Logical 0-1 laws

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We say K has a 0-1 *law* if for every $\mathcal{L}$-sentence ϕ,

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If K has a 0-1 law, then $T_{as}(K)$ is complete.

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Question

Does $M_r := \bigcup_{n \in \mathbb{N}} M_r(n)$ have a 0-1 law?

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Idea: Precise structure theorem + 0-1 law for $C_r \Rightarrow$ new 0-1 law for M_r.

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Enumeration: $|M_r(n)| = |C_r(n)|(1 + 2^{o(n^2)}) = (\frac{r}{2} + 1)^{\binom{n}{2} + o(n^2)}$.

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0-1 law for complicated family $(M_r(n))$

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  6. hypergraph containers theorem (Balogh-Morris-Samotij, Saxton-Thomason).

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Main Ingredients:

- Hypergraph containers theorem (Balogh-Morris-Samotij, Saxton-Thomason).
- Triangle Removal for $\mathcal{L}$-structures (Aroskar-Cummings).
- Many combinatorics papers which have made the pattern of proof clear. Particularly recent work using the hypergraph containers theorem.

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In the appropriate language, most of the results we want to generalize are for hereditary $\mathcal{L}$-properties.

Enumeration and structure of hereditary properties in the setting of graphs and other combinatorial structures have been studied in combinatorics.

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2. What is the approximate structure of $\bigcup_{n \in \mathbb{N}} \mathcal{H}_n$?

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Notation: $V^{\underline{\ell}} = \{(a_1, \ldots, a_\ell) \in V^\ell : a_i \neq a_j \text{ each } i \neq j\}$.

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$S_r(\mathcal{H})$ is the set of complete, quantifier-free $\mathcal{L}$-types $p(x_1, \ldots, x_r)$ s.t. for each $i \neq j$, $x_i \neq x_j \in p(\bar{x})$ and $p(\bar{x})$ is realized in some element of $\mathcal{H}$.

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- For all $p, q \in S_r(\mathcal{H})$, if $p(x_1, \ldots, x_r) = q(x_{\mu(1)}, \ldots, x_{\mu(r)})$ for some permutation μ of $[r]$, then for all $(a_1, \ldots, a_r) \in V^{\underline{r}}$,

$$M \models R_p(a_1, \ldots, a_r) \text{ if and only if } M \models R_q(a_{\mu(1)}, \ldots, a_{\mu(r)}).$$

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Then $S_r(\mathcal{H}) = \{p(x,y), q(x,y)\}$ and $\mathcal{L}_{\mathcal{H}} = \{R_p(x,y), R_q(x,y)\}$.

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This is basically the same as a definition of Aroskar-Cummings.

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If $\mathcal{H}$ is a hereditary $\mathcal{L}$-property with $\pi(\mathcal{H}) > 1$ and a 0-1 law, then $\mathcal{H}$ has a stability theorem.

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