Amalgamation and Symmetries in the Finite

Martin Otto
TU Darmstadt

Model Theory of Finite and Pseudofinite Structures
Leeds, July 2016

a generic free amalgamation construction

from families

$$\left.\begin{array}{rl} (\mathcal{A}_s)_{s \in S} & \text{of relational structures} \\ (\rho_e \colon \mathcal{A}_s \xrightarrow{\text{part}} \mathcal{A}_{s'})_{e \in E[s,s']} & \text{of partial isomorphisms} \end{array}\right\} (*)$$

construct a natural free amalgam of disjoint copies $(\mathcal{A}_s, w) \simeq \mathcal{A}_s$ tagged by walks w into $s \in S$, with overlap between $(\mathcal{A}_s, w)$ and $(\mathcal{A}_{s'}, w \cdot e)$ according to ρ_e

to obtain structure $\big((\mathcal{A}_s) \otimes \mathrm{I}^*\big) \,/ \approx$

based on multi-sorted monoid I^* of walks in $\mathrm{I} = (S, (E[s,s']))$

- realising the overlap pattern specified in $(*)$,
- free in a universal algebraic sense, and
- typically, and mostly necessarily, infinite

finite analoga for finite data? to be based on what?

NB: the structure I^* of walks in the multigraph $(S, (E[s, s']))$ is a multi-sorted monoid w.r.t. a partial concatenation operation

for $s \in S$, can concatenate (walks to s) $\times$ (walks from s)

explore analogies/challenges in passage

- from global to partial operations:

  free word monoid E^* $\rightsquigarrow$ multi-sorted I^*
  groups $\rightsquigarrow$ groupoids/inverse semigroups

- from global to partial symmetries:

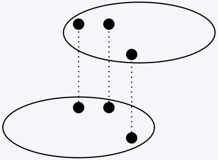
  symmetry groups $\rightsquigarrow$ groupoids/inverse semigroups
  Cayley groups $\rightsquigarrow$ groupoidal analogues

- from graphs to hypergraphs

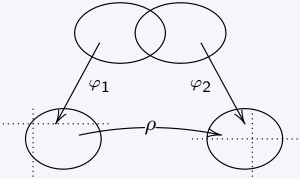
with a view to equally generic but finite constructions

examples of local views with overlaps

- **exploded view of a hypergraph**



- **coordinate charts for manifolds**



- **decomposition and synthesis of graphs, hypergraphs, ...**

- **specifications of guarded $\forall^*\exists^*$ extension properties**

hypergraphs

hypergraph $\mathcal{A} = (A, S)$: $\begin{cases} A \text{ the set of vertices} \\ S \subseteq \mathcal{P}(A) \text{ the set of hyperedges} \end{cases}$

examples:

- **parts (bag structures) of structural decompositions**
- **distinguished (guarded) subsets of relational structures**

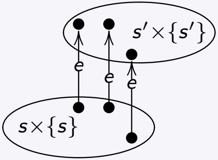
intersection graph $\mathrm{I}(\mathcal{A})$ of $\mathcal{A}$:

$\mathrm{I}(\mathcal{A}) := (S, E)$ where $E = \{(s, s') : s \neq s', s \cap s' \neq \emptyset\}$

records pairwise overlaps between hyperedges $s \in S$

exploded view $\mathrm{H}(\mathcal{A})$ of $\mathcal{A}$ based on $\mathrm{I}(\mathcal{A})$

the disjoint union of the hyperedges $s \in S$
with partial bijections ρ_e for $e = (s, s') \in E$



the hypergraph as a realisation of its overlap pattern

realisation of $\mathrm{H} = \big((\mathcal{A}_s),(\rho_e)\big)$ in general:

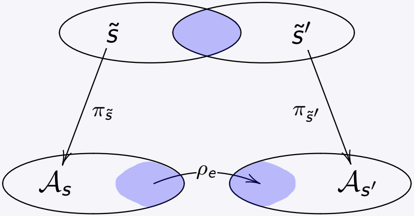
a relational structure $\mathcal{A}$ with an atlas given by superimposed hypergraph structure $(A, \tilde{S})$ of charts $\pi_{\tilde{s}} \colon \mathcal{A}\restriction\tilde{s} \simeq \mathcal{A}_{\pi(\tilde{s})}$ s.t.

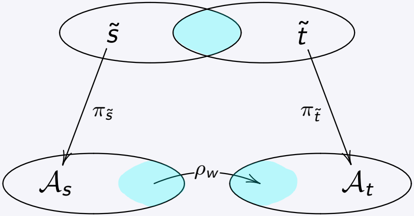
- **locally, all ρ_e-overlaps are realised:**

  each $\pi_{\tilde{s}}^{-1}(\mathcal{A}_s)$ overlaps with some $\pi_{\tilde{s}'}^{-1}(\mathcal{A}_{s'})$ according to ρ_e

- **globally, no incidental overlaps occur:**

  if $\tilde{s} \cap \tilde{t} \neq \emptyset$, then this is due to $\rho_w = \rho_{e_m} \circ \cdots \circ \rho_{e_1}$
  for some *single* walk $w = e_1 \cdots e_m$ from $\pi(\tilde{s})$ to $\pi(\tilde{t})$



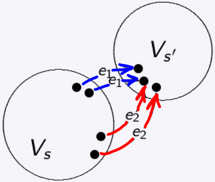


realisations vs. exploded views

any hypergraph $\mathcal{A} = (A, S)$ is a realisation of its exploded view

- **$\mathcal{A}$ is reconstructed from its exploded view $\mathrm{H}(\mathcal{A})$ as a quotient $\mathrm{H}(\mathcal{A})/\!\approx$ w.r.t. $\approx$ induced by the identifications encoded in $\mathrm{H}(\mathcal{A})$**

- in general $\mathrm{H}/\!\approx$ may fail to realise H:
  $\approx$ may even collapse individual s



idea: try partial unfolding in products of H with . . . ?

instead of I^*, seek finite match for $\mathrm{I} = (S, (E[s, s']))$

overview

(I) specification & realisation of amalgamation patterns

(II) reduced products with groupoids (core results)

(III) from local to global symmetries (EPPA)

(IV) applications in modal & guarded logics

the role of groupoids/inverse semigroups

composition structure of partial bijections:

- with partial composition (as a total operation)
  $\rightsquigarrow$ inverse semigroups
- with exact composition (as a partial operation)
  $\rightsquigarrow$ groupoids

groupoids capture local/partial symmetries

. . . just as groups capture global symmetries

(II) reduced products with groupoids

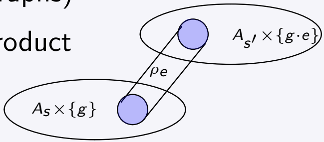
I-groupoid: $\mathbb{G} = \big(G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S}\big)$ with

associative compositions $G_{st} \times G_{tu} \rightarrow G_{su}$,
neutral elements $1_s \in G_{ss}$, inverses, ...
designated generators $(g_e)_{e \in E}$

reduced products as candidate realisations:

$\rightsquigarrow$ $\mathrm{H} \times \mathbb{G}$ natural direct product (of I-graphs)

$\rightsquigarrow$ $\mathrm{H} \otimes \mathbb{G} := (\mathrm{H} \times \mathbb{G})/\approx$ reduced product

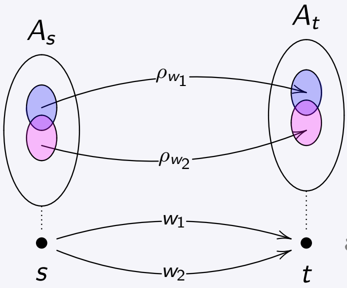


question: when is this a realisation of H?

obstructions to simple realisations:

violations of path independence

- conflicting identifications collapsing individual A_s



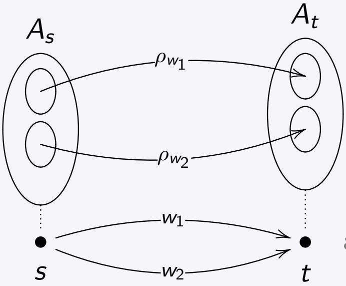
a bad cycle in $\mathbb{G}$?

can be overcome by relatively simple pre-processing

obstructions to simple realisations

violations of path independence/confluence

- incidental multiple overlaps may lead to conflicting identifications at the relational level

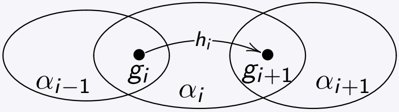


a bad cycle in $\mathbb{G}$?

$\rightsquigarrow$ **need substantial acyclicity conditions on $\mathbb{G}$**

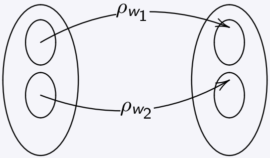
an appropriate notion of acyclicity: control coset cycles

- not just short cycles in the Cayley graph of $\mathbb{G}$, but short cycles of cosets $g\mathbb{G}[\alpha]$ generated by subsets $\alpha \subseteq E$

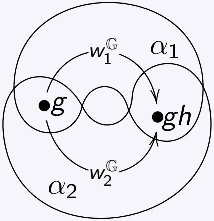


- in particular, need to avoid certain coset cycles of length 2 for relational consistency

$A_s \times \{g\}$



$A_t \times \{gh\}$



$$w_1^{\mathbb{G}} = h = w_2^{\mathbb{G}}$$

in reduced product . . . **. . . in $\mathbb{G}$**

any degree of acyclicity in finite groupoids

theorem (O_13)

for every $\mathsf{N} \in \mathbb{N}$ and incidence pattern $\mathrm{I} = (S, E)$ there are finite I-groupoids $\mathbb{G}$ without coset cycles of length up to N

idea: in an inductive construction generate $\mathbb{G}$ from (semi)group action on amalgamation chains that unfold short cosets cycles

cf. constructions of acyclic Cayley graphs (Alon, Biggs) here lifted to more intricate adaptation for coset cycles

any degree of acyclicity in symmetric realisations

theorem (O_13)

for any overlap specification $\mathbb{H} = \big((\mathcal{A}_s), (\rho_e)\big)$, obtain realisations $\mathbb{H} \otimes \mathbb{G}$ (as reduced products with finite $\mathbb{I}$-groupoids $\mathbb{G}$) that

- have any desired degree of (local/size-bdd) acyclicity
- admit transitive automorphisms in the second factor
- respect all symmetries of the specification $\mathbb{H}$

symmetric realisations

corollary

every finite hypergraph admits, for $\mathsf{N} \in \mathbb{N}$, finite coverings that

- are N-acyclic in the sense that every induced sub-hypergraph on up to N vertices is acyclic (tree decomposable)
- possess a fibre-transitive automorphism group that lifts all symmetries of the given hypergraph

(III) from local to global symmetries

extension property for partial automorphisms (EPPA): how to extend local symmetries to global symmetries

theorem (Herwig 98, extending Hrushovski 92 for graphs)

every finite relational structure $\mathcal{A}$ admits a finite extension $\mathcal{B} \supseteq \mathcal{A}$ s.t. every partial isomorphism in $\mathcal{A}$ lifts to a full automorphism of $\mathcal{B}$

theorem (Herwig–Lascar 00)

same, as a *finite model property* over any class $\mathcal{C}$ defined by finitely many forbidden homomorphisms

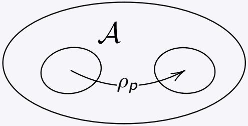
if $\mathcal{A} \in \mathcal{C}_{\mathrm{fin}}$ has any EPPA extension in $\mathcal{C}$ then it also has a finite one in $\mathcal{C}_{\mathrm{fin}}$

new proof of Herwig–Lascar EPPA

through groupoidal realisations of an overlap specification
$\mathrm{H} = \big((\mathcal{A}), (\rho_p)\big)$ for $\mathcal{A} = (A, R)$ and $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{A})$

(i) **the incidence pattern $\mathbf{I}(\mathcal{A}, P)$:**
multigraph on singleton vertex
with a loop $e_p \in E$ for each $p \in P$

(ii) **the overlap specification $\big((\mathcal{A}), (\rho_p)\big)$:**
after pre-processing, $\big((\mathcal{A}_s), (\rho_e)\big)$
turns non-trivially groupoidal (!)



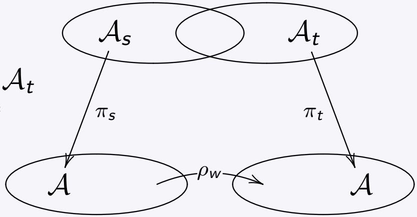
(iii) **symmetric realisations of $\big((\mathcal{A}), (\rho_p)\big)$**
are EPPA extensions

(iv) ***N*-acyclic EPPA extensions are *N*-free:**
admit N-local homomorphisms into every (finite or infinite)
EPPA extension due to their N-local tree-decomposability

the EPPA extensions we get for $(\mathcal{A}, P)$:

finite relational extension structure $\mathcal{B} = (B, R) \supseteq (A, R) = \mathcal{A}$, with superimposed hypergraph structure (B, S), $S \subseteq \mathcal{P}(B)$, and projections $(\pi_s)_{s \in S}$ such that:

- $\mathcal{B} = \bigcup_{s \in S} \mathcal{A}_s$ where $\mathcal{A}_s := \mathcal{B} \restriction s$
- $\big(\pi_s \colon \mathcal{A}_s \simeq \mathcal{A}\big)_{s \in S}$ an atlas for $\mathcal{B}$
- overlaps between charts $\mathcal{A}_s$ and $\mathcal{A}_t$ induced by compositions $w \in \mathrm{P}^*$



- up to any desired size bound, every small substructure of $\mathcal{B}$ is acyclic and covered by $\mathcal{A}$-charts that form a free amalgam **!**

(IV) further applications in modal/guarded logics

- **characterisation theorems** (fmt)
  for the guarded fragment GF and relatives

  using finite coverings of controlled acyclicity

- **characterisation theorems** (fmt & classical)
  for (modal) common knowledge logic

  new, with Felix Canavoi, using (finite) $S5$-frames
  over Cayley groups with controlled coset-acyclicity

- **finite model properties & finite controllability**
  for guarded logics and constraints

  using finite coverings of controlled acyclicity
  and/or Herwig–Lascar extension properties

GF and its characteristic preservation property

the guarded fragment GF (Andréka–van Benthem–Németi 98)

key idea: relativise quantification to guarded clusters

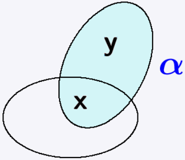
hypergraph $H(\mathcal{A}) = (A, S[\mathcal{A}])$ of guarded subsets $[\boldsymbol{a}]$ for $\boldsymbol{a} \in R^{\mathcal{A}}$

guarded quantification:

$$\exists \boldsymbol{y}\big(\alpha(\boldsymbol{xy}) \wedge \varphi(\boldsymbol{xy})\big)$$

$$\forall \boldsymbol{y}\big(\alpha(\boldsymbol{xy}) \rightarrow \varphi(\boldsymbol{xy})\big)$$

guard atom α: $\mathrm{free}(\varphi) \subseteq \mathrm{var}(\alpha)$



quantification relativised to guarded tuples

ML $\subsetneq$ GF $\subsetneq$ FO

model-theoretic motivation: reflection on ML $\subseteq$ FO in extension from graph-like structures to general relational format

GF and guarded bisimulation

guarded bisimulation $\sim_{\mathbf{g}}$ / $\sim_{\mathbf{g}}^{\ell}$

back&forth equivalence of hypergraphs of guarded subsets
that locally respects relational content, or

Ehrenfeucht–Fraïssé game (pebble game)
on guarded configurations

variations on hypergraph bisimulation patterns

- guarded negation fragment GNF:
  local isomorphisms $\rightsquigarrow$ **local homomorphisms**
  (Barany–ten Cate–Segoufin 11; O_13)

- new route to common knowledge extensions of ML:
  static clusters $\rightsquigarrow$ **clusters generated as cosets**
  (Canavoi–O_ work in progress)

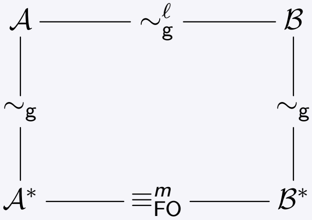
guarded characterisation theorems (fmt & classical)

challenge: expressive completeness results in non-classical settings

theorem (O_10)

$\mathrm{GF} \equiv \mathrm{FO}/{\sim_{\mathrm{g}}}$ and $\mathrm{GF} \equiv_{\mathrm{fin}} \mathrm{FO}/{\sim_{\mathrm{g}}}$

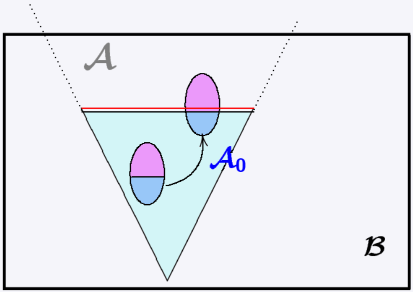
non-classical proof of a compactness property:
$\sim_{\mathrm{g}}$-invariance $\Rightarrow$ $\sim_{\mathrm{g}}^{\ell}$-invariance for some finite ℓ



$\mathcal{A}^*/\mathcal{B}^*$ locally acyclic unfoldings/coverings in reduced products to overcome obsatacles to $\equiv_{\mathrm{FO}}$ that are not controllable by $\sim_{\mathrm{g}}$

finite model property for GF
based on Herwig's EPPA (Grädel 99)

after relational Skolemisation use EPPA to obtain finite model as finite closure of suitable finite substructure of infinite model w.r.t. guarded ∀∃-requirements



variations on guarded fmp

finite controllability (fmp)
for conjunctive queries (homomorphisms)
under guarded constraints

(Rosati 06, Bárány–Gottlob–O_10, Grädel–O_14)

with Herwig–Lascar EPPA instead of Herwig's EPPA:

fmp for GF within classes with forbidden homomorphisms yields
finite controllability of homomorphisms under guarded constraints
counterexamples $\rightsquigarrow$ finite counterexamples

corresponding quantitative results use
locally acyclic unfoldings and coverings
small rather than finite models, complexity-relevant

summary

a generic construction of
highly acyclic finite groupoids

supports

a generic realisation of
finite amalgamation patterns

e.g., towards

- **locally acyclic amalgams in reduced products**
- **locally acyclic hypergraph coverings**
- **non-classical expressive completeness results**
- **finite models properties**
- **extensions of local to global symmetries**

some related references

Bárány–Gottlob–O_(LMCS 2014, arXiv:1309.5822): Querying the guarded fragment

Bárány–ten Cate–O_(VLDB 2012, arXiv:1203.0077): Queries with guarded negation

Grädel–O_(2014): The freedoms of (guarded) bisimulation

Hodkinson–O_(BSL 2003): Finite conformal hypergraph covers and Gaifman cliques in finite structures

Herwig–Lascar(Transactions of the AMS 2000): Extending partial isomorphisms and the profinite topology on free groups

O_(Journal of the ACM 2012): Highly acyclic groups, hypergraph covers and the guarded fragment

O_(arXiv:1404.4599) Finite groupoids, finite coverings & symmetries in finite structures