Simple homogeneous structures

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Homogeneity, simplicity, finiteness and pseudofiniteness

What has this talk to do with finiteness or pseudofiniteness?

- Homogeneous structures are Fraïssé limits of "amalgamation classes" of **finite** structures.
- Many simple homogeneous structures are **pseudofinite** (all stable ones, and probably all **binary** simple homogeneous structures).
- But there are also long standing problems regarding pseudofiniteness of simple homogeneous structures. Is the generic tetrahedron-free 3-hypergraph pseudofinite?
- Some "iconic" homogeneous/homogenizable structures are "probabilistic limits", via 0-1 laws, of **finite** structures.
- ω-categorical structures arising as limits from 0-1 laws tend to be simple. (Is there any "reasonable" exception?)

Homogeneous structures: definitions

Assumption: *the* **vocabulary/signature** *of any structure that we talk about is* **finite** *and* **relational***.*

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1. $\mathcal{M}$ has elimination of quantifiers.
2. Every isomorphism between finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.
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Via the Engeler, Ryll-Nardzewski, Svenonious characterization of ω-categorical theories: *every infinite homogeneous structure has ω-categorical complete theory.*

Classifications of some homogeneous structures

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The following classes of structures, to mention a few, have been *classified*:

1. homogeneous partial orders (Schmerl 1979).
2. homogeneous (undirected) graphs (Gardiner; Golfand, Klin; Sheehan; Lachlan, Woodrow; 1974–1980).
3. homogeneous tournaments (Lachlan 1984).
4. homogeneous directed graphs (Cherlin 1998).
5. infinite homogeneous stable V-structures for any finite relational vocabulary V (Lachlan, Cherlin, Knight... 80ies).
6. homogeneous multipartite graphs (Jenkinson, Truss, Seidel 2012).

A structure is called **binary** if its vocabulary has only *unary and/or binary relation symbols*.

I will assume basic knowledge of simplicity theory, including the notions of **SU-rank and supersimplicity**.

I will speak a lot about **1-based** structures/theories, but actually a formal definition will not be necessary for following the talk.

From hereon 'rank' means 'SU-rank'.

We say that $\mathcal{M}$ has **trivial dependence** if whenever A, B, C are subsets of N^{eq}, where $\mathcal{N}$ is some model of $Th(\mathcal{M})$, and $A \mathop{\not\perp\!\!\!\perp}_C B$, then $A \mathop{\not\perp\!\!\!\perp}_C b$ for some $b \in B$.

Examples

The following structures are **simple, 1-based, with trivial dependence and finite rank** and are **homogeneous** or can be made homogeneous by expanding with a new relation.

1. The *Rado graph* has rank 1.
2. The *generic tetrahedron-free 3-hypergraph* has rank 1.
3. the *line graph of an infinite complete graph* has rank 2 and can be made homogeneous by expanding with a ternary relation.
4. Suppose that $\mathcal{R}$ is a binary random structure. Let $n < \omega$ and let $\mathcal{M}$ be a binary structure with universe $M = R^n$ such that for all $\bar{a}, \bar{a}', \bar{b}, \bar{b}' \in M$ $\mathrm{tp}^{at}_{\mathcal{M}}(\bar{a}, \bar{a}') = \mathrm{tp}^{at}_{\mathcal{M}}(\bar{b}, \bar{b}')$ if and only if $\mathrm{tp}_{\mathcal{R}}(\bar{a}\bar{a}') = \mathrm{tp}_{\mathcal{R}}(\bar{b}\bar{b}')$. Then $\mathcal{M}$ has rank n.
5. For any $n < \omega$ one can construct a structure with rank n by "nesting" n different equivalence relations (with infinitely many infinite classes).
6. The previous two ways of constructing (binary) structures can be combined to produce more complex examples.

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Conclusion: *If $\mathcal{M}$ is homogeneous and 1-based then $\mathcal{M}$ has finite rank and trivial dependence.*

But are all homogeneous simple structures 1-based?

Binary simple homogeneous structures

Theorem (K, 2016): *If $\mathcal{M}$ is* **binary** *simple and homogeneous, then $\mathcal{M}$ is supersimple with finite SU-rank. Also, whenever $a_1, \ldots, a_n \in M$ are pairwise independent over a finite set $A \subseteq M$, then $\{a_1, \ldots, a_n\}$ is independent over A.*

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Palacin noticed that the following result generalizes to simple theories:

Theorem (Goode, 1991): *If $\mathcal{M}$ is stable and the second conclusion in the above theorem holds, then it* **also** *holds when $a_1, \ldots, a_n$ are imaginaries and A a set of imaginaries.*

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By combining the above results one can show that if $\mathcal{M}$ is as in the first theorem then any pregeometry in $\mathcal{M}^{\mathrm{eq}}$ is trivial, and then it follows (by an earlier result of Hart, Kim, Pillay) that $\mathcal{M}$ has trivial dependence, so:

Conclusion: *If $\mathcal{M}$ is* **binary**, *simple and homogeneous, then $\mathcal{M}$ has finite rank and trivial dependence; hence $\mathcal{M}$ is 1-based.*

But I don't know if the conclusion holds for **nonbinary** structures.

Primitive binary simple homogeneous structures

$\mathcal{M}$ is called **primitive** if there is no nontrivial $\emptyset$-definable equivalence relation on M.

Theorem (K, 2016-??): *If $\mathcal{M}$ is* **primitive**, *binary, simple and homogeneous, then $\mathcal{M}$ has rank 1.*

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The conclusion does not hold if the binarity assumption is removed, as witnessed by $\mathcal{H}_2$ later.

Random structures

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$\mathcal{M}$ is a **random structure** if it is *countably infinite, homogeneous* and for every $k = 2, \ldots, r$, there is **no** minimal forbidden structure $\mathcal{A}$ with respect to

$$\mathcal{M} \restriction \{P \in V : \textbf{arity}(P) \leq k\} \quad \text{such that} \quad |A| > k.$$

Example: Rado graph.
Nonexamples: generic triangle-free graph, generic tetrahedron-free 3-hypergraph.

Primitive binary simple homogeneous structures, cont.

Theorem (Aranda Lopez, 2014): *If $\mathcal{M}$ is primitive, binary, simple and homogeneous with rank 1, then $\mathcal{M}$ is a random structure.*

Primitive binary simple homogeneous structures, cont.

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Remark: If the binarity condition is removed then the conclusion is false, as witnessed by $\mathcal{H}_2$, and by the generic tetrahedron-free 3-hypergraph (the later with rank 1).

Primitive binary simple homogeneous structures, cont.

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For $n > 3$, the n-**complete amalgamation property** is a property which allows "stronger amalgamations (of types)" than the *independence theorem*. For $n = 3$ they are equivalent.

Theorem (Palacin, recent): *If $\mathcal{M}$ is 2-transitive (hence primitive), homogeneous, supersimple with maximal arity n and with $(n+1)$-complete amalgamation property, then $\mathcal{M}$ is a random structure (hence of rank 1).*

Binary simple homogeneous structures in general

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Theorem (K, in preparation): *Suppose that $\mathcal{M}$ is binary, simple and homogeneous. Let* **R** *be the (finite) set of all $\emptyset$-definable equivalence relations on M. Then:*

1. *If $a \in M$ and $\mathrm{SU}(a) = s$, then there are $R_1, \ldots, R_s \in \mathbf{R}$, depending only on $\mathrm{tp}(a)$, such that R_{i+1} refines R_i and $\mathrm{SU}(a/a_{R_i}) = s - i$ for all i (or equivalently $\mathrm{SU}(a_{R_{i+1}}/a_{R_i}) = 1$ for all $i < s$).*
2. *Suppose that $a, b, \bar{c} \in M$ and $a \underset{\bar{c}}{\not\smile} b$. Then there is $R \in \mathbf{R}$ such that $a \underset{\bar{c}}{\not\smile} a_R$, $R(a, b)$ (so $a_R \in \mathrm{acl}(b)$) and $a_R \notin \mathrm{acl}(\bar{c})$ (so $\neg R(a, c)$ for every $c \in \mathrm{rng}(\bar{c})$).*
3. *$\mathcal{M}$ is "more or less" random modulo the equivalence relations in* **R**.

1-based nonbinary structures

Observation: *If $\mathcal{M}$ is homogeneous, simple, 1-based and* **2-transitive**, *then $\mathcal{M}$ has rank 1.*

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Proof. If $\mathrm{SU}(\mathcal{M}) > 1$, then, by triviality of dependence, there are distinct $a, b \in M$ such that $a \not\downarrow b$. But there is also $c \in M$ such that $a \downarrow c$. Then $\mathrm{tp}(a, b) \neq \mathrm{tp}(a, c)$. $\square$

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The proof of the previous theorem shows the following:

Proposition: *Suppose that $\mathcal{M}$ is ω-categorical, supersimple with finite rank and with trivial dependence. If $\mathcal{M}$ is* **primitive** *and $\mathrm{SU}(\mathcal{M}) > 1$, then there are distinct $a_i, b_i \in M$, $i = 1, \ldots, 4$, such that $\mathrm{tp}(a_i, a_j) = \mathrm{tp}(b_i, b_j)$ for all i, j and $\mathrm{tp}(a_1, \ldots, a_4) \neq \mathrm{tp}(b_1, \ldots, b_4)$. So if $\mathcal{M}$ is also homogeneous, then it must have some relation symbol of arity 3 or 4.*

The example $\mathcal{H}_k$

Let $k \geq 2$ is a natural number and

let H_k be the set of all k-element subsets of $\mathbb{N}$.

Let $\mathcal{H}_k^- = (H_k, \sim)$ be the graph where, for all $a, b \in H_k$, $a \sim b$ $\iff |a \cap b| = k - 1$.

Let $\mathcal{H}_k$ be the expansion of $\mathcal{H}_k^-$ with a ternary symbol Q such that $\mathcal{H}_k \models Q(a, b, c) \iff |a \cap b \cap c| = k - 1$.

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Claim: (a) *$\mathcal{H}_2$ is homogeneous, primitive and superstable with SU-rank 2 (and 1-based).*
(b) For every $k \geq 3$, some expansion of $\mathcal{H}_k$ by (only) finitely many new relation symbols ought to be homogeneous and primitive, but I have not checked this. It is certainly superstable with rank k.

The example $\hat{\mathcal{H}}_k$

Let $\mathcal{N}_k$ be the structure with vocabulary $\{E\}$, universe $\mathbb{N}$, and where where E is interpreted as an equivalence relation with exactly k classes, all of which are infinite.

Let $\hat{H}_k^-$ be the set of all k-element subsets $A \subseteq \mathbb{N}$ such that, in $\mathcal{N}_k$, if $a, b \in A$ are distinct then $\mathcal{N}_k \models \neg E(a, b)$.

Let $\hat{\mathcal{H}}_k^- = (\hat{H}_k^-, \sim)$ be the graph where for all $a, b \in \hat{H}_k^-$, $a \sim b \iff |a \cap b| = k - 1$.

Let $\hat{\mathcal{H}}_k$ be the expansion of $\hat{\mathcal{H}}_k^-$ with a quaternary symbol Q such that
$\hat{\mathcal{H}}_k \models Q(a_1, a_2, a_3, a_4) \iff a_1 \sim a_2$, $a_3 \sim a_4$ and whenever $b \in a_1 \cap a_2$ there is $c \in a_3 \cap a_4$ such that $\mathcal{N}_k \models E(b, c)$ (and vice versa).

$\hat{\mathcal{H}}_k$ continued...

Claim: *(a)* $\hat{\mathcal{H}}_2$ *is homogeneous and superstable with SU-rank* 2 *(and 1-based).*
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Some thoughts

All structures that I can think of that are simple with SU-rank > 1 and (possibly) primitive and homogeneous are similar to the examples $\mathcal{H}_k$ or $\hat{\mathcal{H}}_k$.

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I think an **unstable** (but simple) variant of $\mathcal{H}_2$ can be obtained like this:

Let $\mathcal{R}$ be the Rado graph. Let E be the set of edges of $\mathcal{R}$. Turn E into a structure $\mathcal{E}$ as follows. For $m = 2, 3$ introduce an m-ary relation symbol for every type of the form $\mathrm{tp}_{\mathcal{R}}(a_1, b_1, \ldots, a_m, b_m)$ where $\{a_i, b_i\}$, $i = 1, \ldots, m$ are edges of $\mathcal{R}$.

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Actually, the "essence" of the proof of the last proposition is to show that some structure *similar* to those discussed must be "embedded" into $\mathcal{M}$ if it is **primitive**, ω-categorical, supersimple with with trivial dependence and finite rank > 1.

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Can we characterize/classify all homogeneous, primitive, simple and 1-based structures? For this we may need a better understanding of simple homogeneous structures of rank 1.

Examples of 3-hypergraphs which are **homogeneous, 2-transitive (hence primitive), simple and 1-based with rank 1**:

(a) The **generic/random 3-hypergraph**.
(b) The **generic tetrahedron-free 3-hypergraph** and other structures constructed similarly (due to recent work of Conant).
(c) The "**parity 3-hypergraph**": Let $\mathcal{G}$ be the Rado graph. Let $\{a, b, c\} \subseteq G$ be a hyperedge iff it contains 1 or 3 edges (from $\mathcal{G}$). Then forget about the underlying Rado graph.

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Note that the parity 3-hypergraph is a reduct of the Rado graph. So one can construct a structure with the given model theoretic properties by taking a "completely random structure", by forbidding some (suitable) substructure(s), or by taking a (suitable) reduct of some (better known) homogeneous simple structure.

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But are there other ways of constructing structures with the properties on top of this page?

Some more problems

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- If 'no', what pregeometries are possible and are all simple (primitive) homogeneous structures supersimple with finite rank?
- The fine structure of (primitive) simple homogeneous structures?
- Generalizations to *rosy* structures (such as the generic triangle-free graph), or other contexts outside of simple structures?

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Thanks for your attention!