Metric ultraproducts of finite metric groups

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Metric groups with bi-invariant metrics

We consider metric groups $(G, \cdot, d)$ which are complete metric spaces where the metric d is bi-invariant and is bounded by 1.

$$d(x_1, x_2) = d(x_1 \cdot y, x_2 \cdot y) = d(y \cdot x_1, y \cdot x_2)$$

$$\text{for all } y \in M.$$

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The **metric** in the ultraproduct $\prod_I (G_i, d_i)/D$ is defined by

$$d((g_i)_I, (g_i')_I) = lim_{i \to D} d_i(g_i, g_i'),$$

i.e. by the rule that the distance between $(g_i)_I$ and $(g_i')_I$ is in the interval $(\varepsilon_1, \varepsilon_2)$ if and only if the set $\{i : d_i(g_i, g_i') \in (\varepsilon_1, \varepsilon_2)\}$ belongs to the ultrafilter D.

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Soficity

Hamming metric: For $g, h \in S_n$ let

$$d_H(g, h) = 1 - \frac{Fix(g^{-1}h)}{n}.$$

An abstract group is **sofic** if it is embeddable into a metric ultraproduct of finite symmetric groups with Hamming metrics.

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Linear soficity, weak soficity,...

For $a, b \in GL_n(\mathbb{C})$ let

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An abstract group is **linear sofic** if it is embeddable into a metric ultraproduct of $\{(GL_n(\mathbb{C}), \rho) : n \in \mathbb{N}\}$ (Arzhantseva, Päunescu). An abstract group is **weakly sofic** if it is embeddable into a metric ultraproduct of finite groups with bi-invariant metrics ≤ 1.

An abstract group is **hyperlinear** if it is embeddable into a metric ultraproduct of $\{(U_n(\mathbb{C}), \frac{1}{2} d_{HS}) : n \in \mathbb{N}\}$, where d_{HS} is the normalised Hilbert-Schmidt metric (i.e. the standard l^2 distance between matrices).

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Metric sofic groups. Let $\mathcal{G}_{sof}$ be the class of complete metric groups with bi-invariant metrics of diameter 1, which are embeddable as closed metric subgroups into a metric ultraproduct of finite symmetric groups with Hamming metrics.

The set of all abstract sofic groups consists of all discrete structures of the class $\mathcal{G}_{sof}$.

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Let us define:

$\mathcal{G}_{hyplin}$ = **hyperlinear metric groups** = metric groups with bi-invariant metrics of diameter 1, which are embeddable as closed subgroups into a metric ultraproduct of all $(U(n), \frac{1}{2}d_{HS})$, $n \in \mathbb{N}$.

$\mathcal{G}_{l.sof}$ = **linear sofic metric groups** = ...

$(GL_n(\mathbb{C}), \rho)$, $n \in \mathbb{N}$.

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Relations, the metric case

$$\mathcal{G}_{sof} \subseteq \mathcal{G}_{w.sof} \subseteq \mathcal{G}$$

$$\mathcal{G}_{l.sof} \subseteq \mathcal{G}_{w.sof}$$

$\mathcal{G}_{sof}$, $\mathcal{G}_{l.sof}$, $\mathcal{G}_{hyplin}$, $\mathcal{G}_{w.sof}$ are axiomatisable!

Let $\mathcal{C}$ be a class of continuous metric structures.
(For exmple let $\mathcal{C}$ be a class of mtrc grps as above, i.e. $\mathcal{C} \subset \mathcal{G}$.)

Theorem

The class $\mathcal{C}$ is **axiomatisable** *in continuous logic* **by** $Th^c_{sup}(\mathcal{C})$ *if an only if it is closed under metric isomorphisms, ultraproducts and taking substructures.*

Corollary

The classes $\mathcal{G}_{sof}$, $\mathcal{G}_{l.sof}$, $\mathcal{G}_{hyplin}$, $\mathcal{G}_{w.sof}$ are sup-axiomatisable (i.e. by its theories Th^c_{sup}).

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$\mathcal{G}_{sof} \neq \mathcal{G}_{w.sof}$.

Let p be a prime number ≥ 13. Let us consider the cyclic group $\mathbb{Z}(p)$ with respect to so called Lee norm and Lee distances:

$$l_{Lee}(a) = \frac{2min(a, p-a)}{p-1} \ , \ d_{Lee}(a,b) = l_{Lee}(a-b).$$

(V.Batagelj , *J.Combin.Inform. System Sci.* 20(1995), no. 1 - 4, 243 - 252.)

Theorem

The metric group $(\mathbb{Z}(p), d_{Lee})$ does not belong to the classes $\mathcal{G}_{sof} \cup \mathcal{G}_{l.sof} \cup \mathcal{G}_{hyplin}$.

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Reduction to the case of discrete structures

If two classes $\mathcal{K}_1$, $\mathcal{K}_2$ from the collection

$$\{\mathcal{G}, \mathcal{G}_{sof}, \mathcal{G}_{w.sof}, \mathcal{G}_{hyplin}, \mathcal{G}_{l.sof}\}$$

have the same countable discrete structures, then $\mathcal{K}_1 = \mathcal{K}_2$.

Indeed if $\mathcal{K}_1$ and $\mathcal{K}_2$ have the same discrete structures then they are generated as axiomatizable classes by the same set of structures.

Theorem

Let (G, d) be a bi-invariant metric group so that $d \leq 1$. Then the following statements hold.
1. (G, d) is a closed subgroup of a metric ultraproduct of discrete bi-invriant metric groups.
2. If (G, d) is hyperlinear then (G, d) is a closed subgroup of a metric ultraproduct of discrete bi-invriant metric groups which are hyperlinear.

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Weakly sofic metric groups

M.Doucha:
any member of $\mathcal{G}$ is a closed subgroup of a metric ultraproduct of finitely generated free groups with discrete bi-invriant metrics ≤ 1.

Problem: $\mathcal{G}_{w.sof} = \mathcal{G}$?

It is enough to prove that any finitely generated free group with a discrete bi-invariant metric ≤ 1 is weakly sofic,

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Let $(G, d) \in \mathcal{G}$ and $\varepsilon \in [0, 1]$. Let:

$$d_\varepsilon(g, h) = \frac{d(g, h) + \varepsilon}{1 + \varepsilon}, \text{ for } g \neq h.$$

We call d_ε the ε-shift of d.

(G, d_ε) is discrete.

Theorem

If (G, d) is a hyperlinear (resp. sofic) metric group then (G, d_ε) is a hyperlinear (resp. sofic) metric group too.

There is a version of this theorem for linear soficity, where for $g \neq h$,

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Linear combinations

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Let d_1 and d_2 be bi-invariant metrics of G so that $(G, d_1), (G, d_2) \in \mathcal{G}_{sof}$ (resp. $\mathcal{G}_{l.sof}$, $\mathcal{G}_{hyplin}$, $\mathcal{G}_{w.sof}$) and $q_1, q_2 \in \mathbb{Q}$ satisfy $q_1 + q_2 = 1$.
Then $(G, q_1 \cdot d_1 + q_2 \cdot d_2) \in \mathcal{G}_{sof}$ (resp. $\mathcal{G}_{l.sof}$, $\mathcal{G}_{hyplin}$, $\mathcal{G}_{w.sof}$).

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Residually finite metric groups

Let Γ be an inverse limit of an inverse system $\mathcal{C}$:

$$H_1 \leftarrow H_2 \leftarrow ... \leftarrow H_i \leftarrow ...,$$

where Γ canonically maps onto H_i by a homomorphism π_i . When $\mathcal{C}$ consist of finite metric groups (H_i, d_i), Γ is in $\mathcal{G}$ with respect to the metric:

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When $(G,d) < (\Gamma, d_{\mathcal{C}})$, we say that (G,d) is **residually finite**,

Theorem

Let (G,d) be a residually finite metric group with respect to an inverse system of finite metric groups which belong to the class $\mathcal{G}_{sof}$ (resp. $\mathcal{G}_{l.sof}$, ..). Then G also belongs to $\mathcal{G}_{sof}$ (resp. $\mathcal{G}_{l.sof}$,..).

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Is $\mathcal{G}_{sof} = \mathcal{G}_{hyplin}$? Is $\mathcal{G}_{sof} = \mathcal{G}_{l.sof}$? Is $\mathcal{G}_{sof} \subseteq \mathcal{G}_{hyplin}$? Is $\mathcal{G}_{sof} \subseteq \mathcal{G}_{l.sof}$?

Identifying permutations of S_n with permutation matrices over $\mathbb{C}$ we obtain:

$$d_H(\sigma,\tau) = \frac{1}{2}(d_{HS}(\sigma,\tau))^2 \text{ and}$$

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Changing metrics

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Problem: Describe functions f so that

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The last case is open even for $f(x) = \frac{x}{x+1}$.

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