Extending Isometries in Metric Spaces with Forbidden Subspaces

Gabriel Conant
University of Notre Dame

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Let $\mathcal{L}$ be a language. Given $\mathcal{L}$-structures A and B, we say B is **symmetric over** A if:

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Definition

A class $\mathcal{K}$ of finite $\mathcal{L}$-structures has the **Hrushovski property** if for all $A \in \mathcal{K}$ there is a $B \in \mathcal{K}$ which is symmetric over A.

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Theorem (Herwig-Lascar 2000)

Let $\mathcal{L}$ be a finite relational language and $\mathcal{F}$ a finite class of finite $\mathcal{L}$-structures. For any finite $\mathcal{F}$-free $\mathcal{L}$-structure A, if there is an $\mathcal{F}$-free $\mathcal{L}$-structure which is symmetric over A, then there is a finite $\mathcal{F}$-free $\mathcal{L}$-structure which is symmetric over A.

Caution: "$\mathcal{F}$-free $\mathcal{L}$-structures" may not always coincide with structures of the desired type. E.g., if $\mathcal{L} = \{E\}$ is the language of graphs compare "K_n-free $\mathcal{L}$-structure" with "K_n-free graph".

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- Let $\mathcal{L} = \{d_s(x, y) : s \in S\}$.
- Let $\mathcal{F}$ be the class of "bad cycles" $(x_1, \ldots, x_n)$ where

$$d(x_1, x_n) > d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n),$$

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- Interpret metric spaces as $\mathcal{F}$-free $\mathcal{L}$-structures.
- Tricky part: Extract a metric space from an $\mathcal{F}$-free $\mathcal{L}$-structure.

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(4) A distance monoid is **semi-archimedean** if, for all $r, s \in R^{>0}$, if $nr < s$ for all $n > 0$ then $r \oplus s = s$.

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(4) Delhommé, Laflamme, Pouzet, Sauer: Fix $R \subseteq \mathbb{N}$, with $0 \in R$. Define the operation

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Each of these is an amalgamation class.

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- Induction step: extend isometries by hand, using semi-archimedean assumption to make things coherent.

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- "Certain technical conditions"
  - (i) Any $\mathcal{F}$-free $\mathcal{R}$-metric space also omits all path extensions of the structures in $\mathcal{F}$.
  - (ii) The maximal distance in $\mathcal{R}$ does not occur in any structure in $\mathcal{F}$. (Ensures $\mathcal{F}^*$ is still finite.)

Examples

Corollary

Let $\mathcal{R}$ be a finite archimedean distance monoid and $\mathcal{F}$ a finite class of finite $\mathcal{R}$-metric spaces such that:

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  (*ii*) Methods work for $1 \leq K_1 \leq \delta$ and all other parameters unchanged.

(3) The technical conditions are artifacts of amalgamating metric spaces with the minimal path metric. There are other methods of amalgamating–can the technical conditions be adapted for these to include more examples?

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