Fractional Helly property in model theory

Artem Chernikov

UCLA

Model theory of finite and pseudofinite structures

Leeds, UK, Jul 27, 2016

Helly's theorem

Theorem

[Helly, 1913] Let $S_1, \ldots, S_n$ be convex subsets of $\mathbb{R}^d$, with $n > d$. If the intersection of every $d+1$ of these sets is non-empty, then the intersection of the whole collection $\bigcap_{i=1}^{n} S_i$ is non-empty.

Fractional Helly's theorem

Theorem

[Katchalski, Liu, 1979] Fix dimension $d \geq 1$. The for every $\alpha \in (0,1]$ there exists $\beta = \beta(\alpha, d) \in (0,1]$ such that the following holds:
If $S_1, \ldots, S_n$ are convex sets in $\mathbb{R}^d$, $n \geq d+1$, such that $\bigcap_{i \in I} S_i \neq \emptyset$ for at least $\alpha \binom{n}{d+1}$ sets $I \in \binom{[n]}{d+1}$, then there is some $J \subseteq [n]$ such that $|J| \geq \beta n$ and $\bigcap_{i \in J} S_i \neq \emptyset$.

FHP for set systems

- Let now $(X, \mathcal{F})$ be an arbitrary set system (i.e. X is a set and $\mathcal{F}$ is a family of subsets of X).

Definition

We say that $\mathcal{F}$ satisfies the *fractional Helly property*, or FHP, if there is some $d \in \mathbb{N}$ such that for every $\alpha \in (0,1]$ there exists $\beta \in (0,1]$ satisfying the following:
For any $S_1, \ldots, S_n \in \mathcal{F}$ such that $\bigcap_{i \in I} S_i \neq \emptyset$ for at least $\alpha\binom{n}{d}$ sets $I \in \binom{[n]}{d}$, then there is some $J \subseteq [n]$ such that $|J| \geq \beta n$ and $\bigcap_{i \in J} S_i \neq \emptyset$.
The minimal d for which this holds is the *fractional Helly number* for $\mathcal{F}$ (if this holds for d, then also holds for any $d' \geq d$).

FHP for formulas

- Let T be a complete first-order theory in a language L, $M \models T$ and $\phi(x, y) \in L$ a formula.
- We associate with it a definable family $\mathcal{F}_\phi = \{\phi(M, b) : b \in M_y\}$ of subsets of M_x.

Definition

We say that $\phi(x, y)$ has FHP if the family $\mathcal{F}_\phi$ has FHP (and the fractional Helly number of ϕ is the fractional Helly number of $\mathcal{F}_\phi$).
T has FHP if every formula has FHP.
(Note: properties of the theory, rather than of the specific model M).

FHP and Shelah's classification

- FHP implies NTP_2.
- FHP implies low.
- [Matousek, 2003] NIP implies FHP. More precisely, if the VC-codensity of $(X, \mathcal{F})$ is $< k$, then k is a fractional Helly number for $\mathcal{F}$.
- If all formulas $\phi(x, y)$ with $|x| = 1$ have FHP, then T has FHP.

FHP relatively to a class of measures

- Recall: a Keisler measure μ on M_y is a finitely additive probability measure on the Boolean algebra of definable subsets of M_y.
- Let $\mathfrak{M}$ be a class of measures on M_y such that for every $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in \mathfrak{M}$ we fix a certain product measure μ^* on $M_{n \times y}$. If $\mu_i = \mu$ for $i = 1, \ldots, n$ we denote μ^* by $\mu^{(n)}$.

Definition

We say that $\phi(x, y)$ has *FHP relatively to* $\mathfrak{M}$ if there is some $d \in \mathbb{N}$ such that for any $\alpha > 0$ there is $\beta > 0$ satisfying: for any $\mu \in \mathfrak{M}$, if $\mu^{(d)}\left(\exists x \bigwedge_{i=1}^{d} \phi(x, y_i)\right) \geq \alpha$ then there is some $a \in M_x$ with $\mu(\phi(a, y)) \geq \beta$.

- Note: FHP now corresponds to "FHP relatively to the class $\mathfrak{M}_{\mathrm{fin}}$" of measures concentrated on finite sets.

Generically stable measures in NIP, 1

Definition

A Keisler measure μ on M_x is *generically stable* if for every formula $\phi(x, y) \in L$ and $\varepsilon > 0$ there are some $a_1, \ldots, a_m \in M_x$ (possibly with repetitions) such that

$$\left| \mu(\phi(x, b)) - \frac{|\{i : a_i \in \phi(M, b)\}|}{m} \right| < \varepsilon$$

for **every** $b \in M_y$.

- In other words, the *VC-theorem* holds for μ.
- Examples of generically stable measures:
  - A measure concentrated on a finite set,
  - In an *o*-minimal M, Lebesgue measure on $[0, 1]$ (restricted to definable sets),
  - In $\mathbb{Q}_p$, additive Haar measure on the compact ball $\mathbb{Z}_p$.
- The type at $+\infty$ in $(\mathbb{R}, +, \times, <, 0, 1)$ is not generically stable.

Generically stable measures in NIP, 2

- Assume we are given a definable relation $E(x, y) \in \mathrm{Def}(M_{xy})$.
- Let μ and ν be Keisler measures on M_x and M_y, respectively.
- Note that $\mathrm{Def}(M_{xy}) \neq \mathrm{Def}(M_x) \times \mathrm{Def}(M_y)$, and E may not be $\mu \times \nu$-measurable.
- In general, there are many ways to extend the product measure $\mu \times \nu$ to a measure ω on $\mathrm{Def}(M_{xy})$.
- For generically stable measures, we have a canonical choice.

Definition. Given generically stable measures μ, ν, on M_x, M_y respectively, we define a measure $\mu \otimes \nu$ on M_{xy} by

$$\mu \otimes \nu (E(x, y)) = \int_{M_x} \left(\int_{M_y} \mathbf{1}_E(x, y)\, d\nu \right) d\mu.$$

- By generic stability, it is well-defined and we have a Fubini condition: $\mu \otimes \nu = \nu \otimes \mu$.

FHP relatively to generically stable measures

Theorem

If FHP holds relatively to the class $\mathfrak{M}_{fin}$ of measures concentrated on finite sets (e.g. T is NIP), then FHP holds relatively to the class $\mathfrak{M}_{g.s.}$ of generically stable measures (with $\otimes$).

- In fact, [Hrushovski, Pillay, Simon, "A note on generically stable measures and fsg groups"] prove an equivalent statement in the NIP case (equivalent using that the class of generically stable measures in NIP is closed under ultraproducts).

Colorful version

- Actually, no reason to fix only one measure. We have (essentially in [Pillay, "Weight and measure in NIP theories"]):

Theorem

Let T be NIP, such that dp-rank("$x = x$") $\leq d$. Then for any formulas $\phi_1(x, y_1), \ldots, \phi_{d+1}(x, y_{d+1}) \in L$ and $\alpha > 0$ there is some $\beta > 0$ such that:
if μ_i is a generically stable measure on M_{y_i}, $i = 1, \ldots, d+1$, and $\mu_1 \otimes \ldots \otimes \mu_{d+1}\left(\exists x \bigwedge_{i=1}^{d+1} \phi_i(x, y_i)\right) \geq \alpha$ then there is some $a \in M_x$ and $1 \leq i \leq d+1$ such that $\mu_i(\phi_i(a, y_i)) \geq \beta$.

- This corresponds to the so called "colorful fractional Helly property" for NIP families (was known in combinatorics for convex sets, due to [Barany et. al., 2014]).

Corollary

The fractional Helly number of $\phi(x, y)$ is at most dp-rank($x = x$) $+ 1$ (by the Theorem for $\phi_i(x, y_i) = \phi(x, y)$ and $\mu_i = \mu$).

FHP in MS-measurable structures, 1

Definition

[Macpherson, Steinhorn, 2008] An L-structure M is *MS-measurable* if for every non-empty set $X \subseteq M^n$ definable with parameters, we have a pair $(\dim(X), \text{meas}(X))$ with $\dim(X) \in \mathbb{N}$, $\dim(X) \leq n$ and $\text{meas}(X) \in \mathbb{R}_{>0}$ satisfying some strong definability properties and a **Fubini** condition.

Example

Let $M = \prod_{p \in P} \mathbb{F}_p / \mathcal{U}$ be an ultraproduct of finite fields, P a set of primes, $\mathcal{U}$ a non-principal ultrafilter on P.
Let $X \subseteq M^n$ be a definable set, $X = \prod_{p \in P} X_p / \mathcal{U}$ for some $X_p \subseteq \mathbb{F}_p^n$. Then $(\dim(X), \text{meas}(X)) = (d, \alpha)$ if

$$|X_p| \approx \alpha p^d$$

for $\mathcal{U}$-many p.

FHP in MS-measurable structures, 2

- For any definable set $B \subseteq M_y$, we have a Keisler measure μ_B concentrated on B and defined by

$$\mu_B(X) = \begin{cases} \frac{\text{meas}(X \cap B)}{\text{meas}(B)} & \text{if } \dim(X \cap B) = \dim(B), \\ 0 & \text{if } \dim(X \cap B) < \dim(B) \end{cases}$$

  for all definable $X \subseteq M_y$.
- Let $\mathfrak{M}_y = \{\mu_B : B \subseteq M_y \text{ definable}\}$, and we take $\mu_{B_1} \otimes \mu_{B_2} := \mu_{B_1 \times B_2}$ (given by the same double integral as in generically stable measures).

Theorem

In an MS-measurable theory, $\phi(x, y)$ satisfies FHP relatively to the class of measures $\mathfrak{M}_y = \{\mu_B : B \subseteq M_y \text{ definable}\}$.

- In particular, we get that MS-measurable implies FHP (as in particular μ_B with B finite is the counting measure concentrated on B), and the fractional Helly number of $\phi(x, y)$ is at most $\max\{\dim \phi(x, b) : b \in M_y\} + 1$.

FHP for definable families in finite fields

- Since every ultraproduct of finite fields is MS-measurable, it follows that definable families in large finite fields satisfy the fractional Helly property.

Corollary

For every formula $\phi(x, y)$ in the ring language there is some fractional Helly number d such that for every $\alpha > 0$ there is $\beta > 0$ satisfying the following.
For each $\psi(y, z)$, for all (but finitely many) finite fields F and all $c \in F$ we have: if

$$\left| \left\{ S \in \psi(F, c)^d \right\} : \bigcap_{b \in S} \phi(F, b) \neq \emptyset \right| \geq \alpha \left| \psi(F, c) \right|^d ,$$

then there is some $a \in F$ such that
$|\phi(a, F) \cap \psi(F, c)| \geq \beta |\psi(F, c)|$.

Ultraproducts of p-adics

- For each prime p, the field $\mathbb{Q}_p$ is NIP, so satisfies FHP relatively to generically stable measures.
- Problem: For a fixed formula $\phi(x, y)$ in the ring language, can I choose d (and β) independently of p, for the Haar measure on $\mathbb{Q}_p$?
- **Conjecture**. Yes (should follow from the Fubini theorem for motivic integration).
- At least I have:

Theorem

Let $M = \prod_{p \in P} \mathbb{Q}_p / \mathcal{U}$ for P a set of primes, $\mathcal{U}$ a non-principal ultrafilter on P. Then $T = \mathrm{Th}(M)$ satisfies FHP (i.e. for measures concentrated on finite sets).

- Note: T is neither NIP nor simple.

Small sets in model theory

- Let $\phi(x, b) \in L(\mathbb{M})$ and M a small model of T. When is the set that it defines "small"? There are several natural notions.

Definition

[Shelah] $\phi(x, b)$ *divides over M* if there is an infinite sequence $(b_i : i \in \omega)$ and $k \in \mathbb{N}$ such that $b_i \equiv_M b$ and for any $i_1 < \ldots < i_k \in \omega$ we have $\bigcap_{j=1}^{k} \phi\left(M, b_{i_j}\right) = \emptyset$.
$\phi(x, b)$ *forks over M* if it belongs to the ideal generated by the formulas dividing over M (in the Boolean algebra of definable sets in $\mathbb{M}_x$).

- [C., Kaplan, 2012] If T is NTP_2, then the set of formulas dividing over M is already an ideal ("forking = dividing").

Definition

We say that $\phi(x, b)$ is *universally null over M* if $\mu(\phi(x, b)) = 0$ for **all** $\mathrm{Aut}(\mathbb{M}/M)$-invariant global Keisler measures.

- Sets universally null over M also form an ideal.

Comparing these two ideals

- In any theory, if $\phi(x, b)$ forks over M then it is universally null over M.
- In NIP, the converse is also true, so these ideals coincide (every formula non-forking over M extends to a global M-invariant type).
- [Hrushovski] There are simple theories in which these two ideals are different.
- Problem outside of NIP: need to find invariant measures without any kind of (Borel-)definability of types.

Conjecture . Let T be FHP. Then $\phi(x, b)$ forks over M iff $\phi(x, b)$ is universally null.

- (can show under FHP with some additional definability assumptions, e.g. for MS-measurable classes)

Directions / work in progress

1. Proving the conjecture :). In fact, this result should more or less characterize FHP.
2. The theory developed in [C., Simon "Definably amenable NIP groups"] should generalize to definably amenable FHP groups. (e.g. Petrykowski's conjecture, equivalence of different notions of genericity, etc.)
3. Subadditivity of burden in FHP theories.
4. ULCFS conjecture for FHP theories.