Metrically Homogeneous Graphs: A complete Census?

Gregory Cherlin

Wednesday, July 27
Finite/Pseudofinite
Outline

- History
- Census reports; conjecture
- Evidence
- Infinite Diameter
U est homogène en ce sens que, les ensembles finis et congruents A et B (situés dans U) étant quelconques, il existe une représentation isométrique de U sur lui-même transformant A en B. —(Urysohn 1927 CRAS)

metric congruence $\implies$ Klein congruence

(Erlangen program: the automorphism group determines the language)
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metric congruence $\Rightarrow$ Klein congruence
(Erlangen program: the automorphism group determines the language)

We require this for labeled sets.
History: Henson to Moss

Henson 1971: A family of countable homogeneous graphs
Woodrow 1979: There are four countable ultrahomogeneous graphs without triangles
Lachlan/Woodrow 1980: Countable ultrahomogeneous undirected graphs
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Larry Moss 1992: Distanced graphs

\[\text{. . . whether every distance homogeneous graph is distance finite. In the countable case, an answer to this question might be a step towards a classification of the distance homogeneous graphs. [ . . . cf. Cameron, Lachlan/Woodrow]}\]

MR1169148: As the author notes, the problem of characterizing all the countable homogeneous graphs for the expanded language remains open.
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Cameron 1977: *6-Transitive graphs* (finite)

Macpherson 1982: *Infinite distance transitive graphs of finite valency* (locally finite)

\(T_{m,n}\): tree-like, each vertex belongs to \(m n\)-cliques.
Interlude: Finite metrically homogeneous graphs

Finite primitive homogeneous graphs: $C_5$, $K_3 \otimes K_3$

$C_n$: metrically homogeneous of diameter $\delta = \lfloor n/2 \rfloor$
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$K_n \otimes K_n$: 4-ary!

$2 K_2$
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$C_n$: metrically homogeneous of diameter $\delta = \lfloor n/2 \rfloor$
Other finite: antipodal double covers of $C_5$, $K_3 \otimes K_3$, $I_n$
(diameter 3)

\[d(a, b') = d(a, b)' = 3 - d(a, b)\]
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Double cover of $C_5$:

**Classification**

- Diameter $\leq 2$
- Antipodal double cover of $I_n$, $C_5$, $K_3 \otimes K_3$
- $C_n$
(General) History: Nešetřil to KPT

Nešetřil: 1989 *For graphs* . . .; 2005 *Ramsey classes and homogeneous structures* (Ramsey implies amalgamation)

KPT 2005 *Fraïssé limits, Ramsey theory, and topological dynamics* . . .
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History

Census takers

Evidence

infinite diameter
Cameron’s Census

Cameron 1998: A census of infinite distance-transitive graphs

Not even the countable metrically homogeneous graphs have been determined. 😞
Cameron’s Census

Cameron 1998: A census of infinite distance-transitive graphs

Not even the countable metrically homogeneous graphs have been determined. 😞

Census

- Locally Finite: known
- Diameter \( \leq 2 \): known
- \( \Gamma^\delta \): Urysohn graph of diameter \( \delta \)
- Bipartite Urysohn graph of diameter \( \delta \) (all triangles have even perimeter)
- Henson variation (\( K_n \)-free)
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*This construction is similar, but not identical, to one due to Komjath et al. . . . , who constructed a countable universal graph omitting odd cycles up to some fixed length. No doubt, further such variations are possible.*
Interlude: Komjáth, Mekler, Pach

KMP 1988 Some universal graphs

Theorem

1. For any $K$, there is a universal countable graph among all graphs with no odd cycle of length less than $2K + 1$.

2. For any $C$, there is a universal countable graph among all graphs with no odd cycle of length greater than $C$. 
My two censuses

**FIRST CENSUS (2009)**
- Exceptions: finite, diameter $\leq 2$, or tree-like
- Generic type: constraints specified by
  - diameter $\delta$
  - KMP parameters $K$, $C$ (for triangles)
  - generalized Henson constraints $(1, \delta)$-spaces, or antipodal Henson constraints ($\delta \geq 4$)

Abstract form: known exceptions + $(A = A_3 \cap A_H)$
My two censuses

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Abstract form: known exceptions $+$ $(\mathcal{A} = \mathcal{A}_3 \cap \mathcal{A}_H)$

**SECOND CENSUS (2010)—AND CONJECTURE**
- Exceptions: finite, diameter $\leq 2$, or tree-like
- Generic type: constraints specified by
  - diameter $\delta$
  - KMP parameters $K_1$, $K_2$, $C_0$, $C_1$ (for triangles)
  - generalized Henson constraints $(1, \delta)$-spaces, or
    antipodal Henson constraints ($\delta \geq 4$)

Abstract form: the same
**Triangle constraints of type** $K_1, K_2, C_0, C_1$

$p = \text{perimeter of } \Delta, \ |\Delta| = \text{diameter},$

FORBIDDEN TRIANGLES

<table>
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*Remark* Uniformly definable in Presburger arithmetic. Hence for any $k$, $k$-amalgamation is given by a quantifier-free condition on the numerical parameters.
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Remark

*Uniformly definable in Presburger arithmetic. Hence for any $k$, $k$-amalgamation is given by a quantifier-free condition on the numerical parameters.*
A 3-constrained class of metric graphs associated with parameters \((\delta, K_1, K_2, C_0, C_1)\) is an amalgamation class if and only if it has 5-amalgamation.
3-constrained amalgamation class

Theorem

A 3-constrained class of metric graphs associated with parameters \((\delta, K_1, K_2, C_0, C_1)\) is an amalgamation class if and only if it has 5-amalgamation.

What are the numerical conditions? There are three possibilities.
3-constrained amalgamation class

Theorem

A 3-constrained class of metric graphs associated with parameters $(\delta, K_1, K_2, C_0, C_1)$ is an amalgamation class if and only if it has 5-amalgamation.

What are the numerical conditions? There are three possibilities.

1. Bipartite: $K_1 = \infty$

   $K_2 = 0$, $C_1 = 2\delta + 1$
Theorem

A 3-constrained class of metric graphs associated with parameters \((\delta, K_1, K_2, C_0, C_1)\) is an amalgamation class if and only if it has 5-amalgamation.

What are the numerical conditions? There are three possibilities.

II. Low: \(K_1 < \infty, C = \min(C_0, C_1) \leq 2\delta + K_1\)

- \(C = 2K_1 + 2K_2 + 1\);
- \(K_1 + K_2 \geq \delta\);
- \(K_1 + 2K_2 \leq 2\delta - 1\)

\((IIA)\) \(C' = C + 1\) or

\((IIB)\) \(C' > C + 1, K_1 = K_2, \text{and } 3K_2 = 2\delta - 1\)
A 3-constrained class of metric graphs associated with parameters \((\delta, K_1, K_2, C_0, C_1)\) is an amalgamation class if and only if it has 5-amalgamation.

What are the numerical conditions? There are three possibilities.

III. High \(K_1 < \infty\), \(C = \min(C_0, C_1) > 2\delta + K_1\)

- \(K_1 + 2K_2 \geq 2\delta - 1\) and \(3K_2 \geq 2\delta\);
- If \(K_1 + 2K_2 = 2\delta - 1\) then \(C \geq 2\delta + K_1 + 2\);
- If \(C' > C + 1\) then \(C \geq 2\delta + K_2\).
Metrically Homogeneous Graphs: A complete Census?

1. History

2. Census takers

3. Evidence

4. Infinite diameter
Unconditional Evidence

**Definition (Generic type)**

A metrically homogeneous graph has **generic type iff**

- $\Gamma_1$ is primitive; and
- For two vertices at distance 2, their common neighbors contain an infinite independent set.
### Definition (Generic type)
- $\Gamma_1$ is primitive; and
- For two vertices at distance 2, their common neighbors contain an infinite independent set.

### Theorem (Unconditional Evidence)
1. Non-generic type are classified
2. All amalgamation classes defined by forbidden triangles and Henson constraints are of known type.
3. (with Amato and Macpherson) The conjecture is valid in diameter 3.
4. (Local analysis, generic type) If $\Gamma_i$ has an edge then it is metrically homogeneous, connected, generic type usually.
Conditional evidence

Theorem (Conditional Evidence)

1. If $\Gamma$ is bipartite and the half-graph $B\Gamma$ is of known type, then $\Gamma$ is known.

2. If $\Gamma$ has infinite diameter and all local subgraphs $\Gamma_i$ which contain an edge are of known type, then $\Gamma$ is of known type.
Conditional evidence

**Theorem (Conditional Evidence)**

1. *if $\Gamma$ is bipartite and the half-graph $B\Gamma$ is of known type, then $\Gamma$ is known.*

2. *If $\Gamma$ has infinite diameter and all local subgraphs $\Gamma_i$ which contain an edge are of known type, then $\Gamma$ is of known type.*

The bipartite case reduces to the case in which $K_1 = 1$. So it would be interesting to treat this case fully and kill two birds with one stone.
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infinite diameter
Infinite diameter: Reduction

Now we discuss the proof of the infinite diameter reduction.

**Theorem (Infinite Diameter)**

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**Target:** \( \Gamma_{K_1,S}^{\infty} \)

\((S = \{K_n\} \text{ or empty})\)
Infinite diameter: Reduction

Now we discuss the proof of the infinite diameter reduction.

**Theorem (Infinite Diameter)**

*If \( \Gamma \) has infinite diameter and all local subgraphs \( \Gamma_i \) which contain an edge are of known type, then \( \Gamma \) is of known type.*

**Target:** \( \Gamma^{\infty}_{K_1, S} \)

\((S = \{K_n\} \text{ or empty})\)

- Reduce to the case \( K_1 < \infty \)
- Show that \( \Gamma_i \) contains an edge for \( i \geq K_1 \) and apply local analysis to conclude.
The infinite diameter bipartite case

**Difficulty**: locally, no edges.
The infinite diameter bipartite case

**Difficulty:** locally, no edges.
But we have a reduction from $\Gamma$ to $B\Gamma$. 

The infinite diameter bipartite case

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Difficulty: diameter does not go down.
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But $K_1 = 1$ (or classified previously).
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O.K. THEN 😊
Infinite diameter, $K_1 < \infty$

**Lemma (Embedding lemma)**

*If A omits triangles of small odd perimeter then A embeds into $\Gamma$.***
Infinite diameter, $K_1 < \infty$

**Lemma (Embedding lemma)**

*If* $A$ *omits triangles of small odd perimeter then* $A$ *embeds into* $\Gamma$.

**Plan of attack.**

- $\Gamma_i$ contains an edge when $i \geq K_1$.
- When $i$ is large, then the diameter of $\Gamma_i$ and the values of $K_2, C_0, C_1$ are all large, and hence do not constrain $A$.
- Prove a local clique lemma (avoid $\Gamma_1$)
- For $i \geq \max(K_1, 2)$, the value $\tilde{K}_1$ of the numerical parameter $K_1$ associated to $\Gamma_i$ is equal to the original $K_1$.

$A$ embeds into $\Gamma_i$ for $i$ large, hence into $\Gamma$. 
The easy bits

---

-the diameter $\tilde{\delta}$ of $\Gamma_i$ is $2i$.

-$\tilde{C}_0, \tilde{C}_1 \geq 2\tilde{\delta}$

-$\tilde{K}_2 \geq \tilde{\delta}/2$
The easy bits

— the diameter $\tilde{\delta}$ of $\Gamma_i$ is $2i$.
— $\tilde{C}_0, \tilde{C}_1 \geq 2\tilde{\delta}$
— $\tilde{K}_2 \geq \tilde{\delta}/2$

What is left?
For $i \geq \max(K_1, 2)$:

- $\Gamma_i$ contains an edge; moreover
- $\Gamma_i$ contains a triangle of type $(K_1, K_1, 1)$
The tricky bit

**Lemma (Main technical lemma)**

For $\Gamma$ of infinite diameter with $K_1 < \infty$, if all the local graphs $\Gamma_i$ which contain an edge are of known type, then for $i \geq \max(K_1, 2)$ they all contain triangles of type $(K_1, K_1, 1)$. 
The tricky bit

Lemma (Main technical lemma)

For $\Gamma$ of infinite diameter with $K_1 < \infty$, if all the local graphs $\Gamma_i$ which contain an edge are of known type, then for $i \geq \max(K_1, 2)$ they all contain triangles of type $(K_1, K_1, 1)$.

- Base $i = K$ (usually, $K_1$): explicit amalgamation
- Induction for $i > K$ (look at $\Gamma_{i+1}(\Gamma_i)$).
Proof of the lemma \( i = K_1 > 1 \)

<table>
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<td>( (d(v, u_1) = K_1 - 1, \ d(v, u_2) = 2) )</td>
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\( d(w, a_1) = K_1 \)
\( d(w, u_1) = K_1 - 1 \)

\( a_1: \Gamma_{K_1} \)
Blow-up

\[ d(v, u_1) = K_1 - 1, \quad d(v, u_2) = 2 \]

Target: \( \Gamma_1(a_1) \)
Theorem (Smith)

An imprimitive metrically homogeneous graph is bipartite or antipodal.
Desiderata

Theorem (Smith)

An imprimitive metrically homogeneous graph is bipartite or antipodal.

Problem (Unfinished business)

Give a conditional reduction of the antipodal case.