Metrically Homogeneous Graphs: A complete Census?

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Finite/Pseudofinite

Outline

- History
- Census reports; conjecture
- Evidence
- Infinite Diameter

History: Klein to Urysohn

> *U est* homogène *en ce sens que, les ensembles finis et congruents A et B (situés dans U) étant quelconques, il existe une représentation isométrique de U sur lui-même transformant A en B.* —*(Urysohn 1927 CRAS)*

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We require this for *labeled sets*.

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Woodrow 1979: *There are four countable ultrahomogeneous graphs without triangles*
Lachlan/Woodrow 1980: *Countable ultrahomogeneous undirected graphs*

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> *. . . whether every distance homogeneous graph is distance finite. In the countable case, an answer to this question might be a step towards a classification of the distance homogeneous graphs. [. . . cf. Cameron, Lachlan/Woodrow]*

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Cameron 1977: *6-Transitive graphs* (finite)
Macpherson 1982: *Infinite distance transitive graphs of finite valency* (locally finite)
$T_{m,n}$: tree-like, each vertex belongs to m n-cliques.

Interlude: Finite metrically homogeneous graphs

Finite primitive homogeneous graphs: C_5, $K_3 \otimes K_3$
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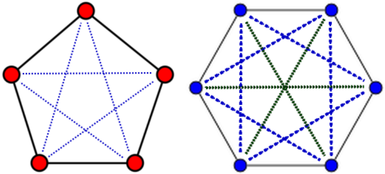

C_5 C_6

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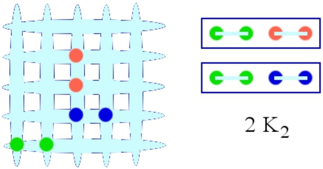
$K_n \otimes K_n$: 4-ary!



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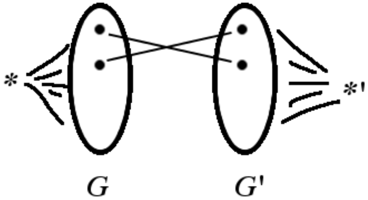
Other finite: antipodal double covers of C_5, $K_3 \otimes K_3$, I_n (diameter 3)



$$d(a, b') = d(a, b)' = 3 - d(a, b)$$

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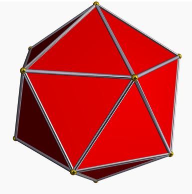

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CLASSIFICATION

- Diameter ≤ 2
- antipodal double cover of I_n, C_5, $K_3 \otimes K_3$
- C_n

(General) History: Nešetřil to KPT

Nešetřil: 1989 *For graphs . . .*; 2005 *Ramsey classes and homogeneous structures* (Ramsey implies amalgamation)

KPT 2005 *Fraïssé limits, Ramsey theory, and topological dynamics . . .*

Cameron's Census

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CENSUS

- Locally Finite: known
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- Γ^δ: Urysohn graph of diameter δ
- Bipartite Urysohn graph of diameter δ (all triangles have even perimeter)
- Henson variation (K_n-free)

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This construction is similar, but not identical, to one due to Komjath et al. . . . , who constructed a countable universal graph omitting odd cycles up to some fixed length. No doubt, further such variations are possible.

Interlude: Komjáth, Mekler, Pach

KMP 1988 *Some universal graphs*

Theorem

1. *For any K, there is a universal countable graph among all graphs with no odd cycle of length less than $2K+1$.*
2. *For any C, there is a universal countable graph among all graphs with no odd cycle of length greater than C.*

My two censuses

FIRST CENSUS (2009)

- Exceptions: finite, diameter ≤ 2, or tree-like
- Generic type: constraints specified by
  - diameter δ
  - KMP parameters K, C (for triangles)
  - generalized Henson constraints $(1, \delta)$-spaces, or antipodal Henson constraints ($\delta \geq 4$)

Abstract form: known exceptions + $(\mathcal{A} = \mathcal{A}_3 \cap \mathcal{A}_H)$

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SECOND CENSUS (2010)—AND CONJECTURE

- Exceptions: finite, diameter ≤ 2, or tree-like
- Generic type: constraints specified by
  - diameter δ
  - KMP parameters K_1, K_2, C_0, C_1 (for triangles)
  - generalized Henson constraints $(1, \delta)$-spaces, or antipodal Henson constraints ($\delta \geq 4$)

Abstract form: the same

Triangle constraints of type K_1, K_2, C_0, C_1

$p =$ perimeter of Δ, $|\Delta| =$ *diameter*,

FORBIDDEN TRIANGLES

	K_1	K_2	C
p odd	$p \leq 2K_1$	$p \geq 2(K_2 + \|\Delta\|)$	$p \geq C_1$
p even			$p \geq C_0$

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Remark

Uniformly definable in Presburger arithmetic.
Hence for any k, k-amalgamation is given by a quantifier-free condition on the numerical parameters.

3-constrained amalgamation class

Theorem

A 3-constrained class of metric graphs associated with parameters $(\delta, K_1, K_2, C_0, C_1)$ is an amalgamation class if and only if it has 5-amalgamation.

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What are the numerical conditions? There are three possibilities.

I. Bipartite: $K_1 = \infty$

$K_2 = 0$, $C_1 = 2\delta + 1$

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What are the numerical conditions? There are three possibilities.

II. Low: $K_1 < \infty$, $C = \min(C_0, C_1) \leq 2\delta + K_1$

- $C = 2K_1 + 2K_2 + 1$;
- $K_1 + K_2 \geq \delta$;
- $K_1 + 2K_2 \leq 2\delta - 1$

(IIA) $C' = C + 1$ or

(IIB) $C' > C + 1$, $K_1 = K_2$, and $3K_2 = 2\delta - 1$

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What are the numerical conditions? There are three possibilities.

III. High $K_1 < \infty$, $C = \min(C_0, C_1) > 2\delta + K_1$

- $K_1 + 2K_2 \geq 2\delta - 1$ and $3K_2 \geq 2\delta$;
- If $K_1 + 2K_2 = 2\delta - 1$ then $C \geq 2\delta + K_1 + 2$;
- If $C' > C + 1$ then $C \geq 2\delta + K_2$.

Unconditional Evidence

Definition (Generic type)

A metrically homogeneous graph has generic type iff

- Γ_1 is primitive; and
- For two vertices at distance 2, their common neighbors contain an infinite independent set.

Unconditional Evidence

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- Γ_1 is primitive; and
- For two vertices at distance 2, their common neighbors contain an infinite independent set.

Theorem (Unconditional Evidence)

1. *Non-generic type are classified*
2. *All amalgamation classes defined by forbidden triangles and Henson constraints are of known type.*
3. *(with Amato and Macpherson) The conjecture is valid in diameter 3.*
4. *(Local analysis, generic type) If Γ_i has an edge then it is metrically homogeneous, connected, generic type usually.*

Conditional evidence

Theorem (Conditional Evidence)

1. *if Γ is bipartite and the half-graph $B\Gamma$ is of known type, then Γ is known.*
2. *If Γ has infinite diameter and all local subgraphs Γ_i which contain an edge are of known type, then Γ is of known type.*

Conditional evidence

Theorem (Conditional Evidence)

1. *if Γ is bipartite and the half-graph $B\Gamma$ is of known type, then Γ is known.*
2. *If Γ has infinite diameter and all local subgraphs Γ_i which contain an edge are of known type, then Γ is of known type.*

The bipartite case reduces to the case in which $K_1 = 1$. So it would be interesting to treat this case fully and kill two birds with one stone.

Infinite diameter: Reduction

Now we discuss the proof of the infinite diameter reduction.

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Target: $\Gamma^{\infty}_{K_1,\mathcal{S}}$!
($\mathcal{S} = \{K_n\}$ or empty)

- Reduce to the case $K_1 < \infty$
- Show that Γ_i contains an edge for $i \geq K_1$ and apply local analysis to conclude.

The infinite diameter bipartite case

Difficulty: locally, no edges.

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O.K. THEN ☺

Infinite diameter, $K_1 < \infty$

Lemma (Embedding lemma)

If A omits triangles of small odd perimeter then A embeds into Γ.

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Plan of attack.

- Γ_i contains an edge when $i \geq K_1$.
- When i is large, then the diameter of Γ_i and the values of K_2, C_0, C_1 are all large, and hence do not constrain A.
- Prove a local clique lemma (avoid Γ_1)
- For $i \geq \max(K_1, 2)$, the value $\tilde{K}_1$ of the numerical parameter K_1 associated to Γ_i is equal to the original K_1.

A embeds into Γ_i for i large, hence into Γ.

The easy bits

—the diameter $\tilde{\delta}$ of Γ_i is $2i$.
—$\tilde{C}_0, \tilde{C}_1 \geq 2\tilde{\delta}$
—$\tilde{K}_2 \geq \tilde{\delta}/2$

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—the diameter $\tilde{\delta}$ of Γ_i is $2i$.
—$\tilde{C}_0, \tilde{C}_1 \geq 2\tilde{\delta}$
—$\tilde{K}_2 \geq \tilde{\delta}/2$

What is left?

For $i \geq \max(K_1, 2)$:

- Γ_i contains an edge; moreover
- Γ_i contains a triangle of type

$$(K_1, K_1, 1)$$

The tricky bit

Lemma (Main technical lemma)

For Γ of infinite diameter with $K_1 < \infty$, if all the local graphs Γ_i which contain an edge are of known type, then for $i \geq \max(K_1, 2)$ they all contain triangles of type $(K_1, K_1, 1)$.

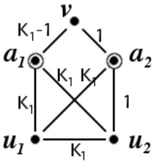
The tricky bit

Lemma (Main technical lemma)

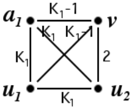
For Γ of infinite diameter with $K_1 < \infty$, if all the local graphs Γ_i which contain an edge are of known type, then for $i \geq \max(K_1, 2)$ they all contain triangles of type $(K_1, K_1, 1)$.

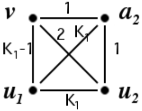
- Base $i = K$ (usually, K_1): explicit amalgamation
- Induction for $i > K$ (look at $\Gamma_{i+1}(\Gamma_i)$).

Proof of the lemma $i = K_1 > 1$

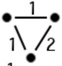


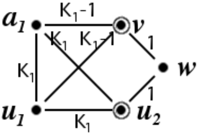
$(d(v, u_1) = K_1 - 1, d(v, u_2) = 2)$
Target: $\Gamma_1(a_1)$



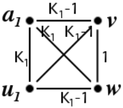


u_1: Γ_{K_1} Γ_{K_1-1}

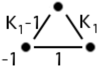


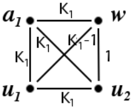


$d(w, a_1) = K_1$
$d(w, u_1) = K_1 - 1$



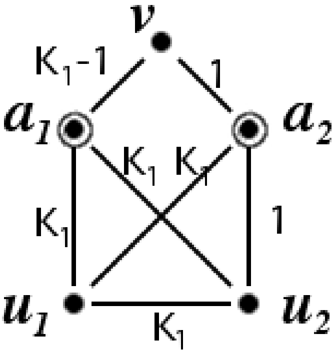
u_1: Γ_{K_1} Γ_{K_1-1}





a_1: Γ_{K_1}

Blow-up



$d(v, u_1) = K_1 - 1$, $d(v, u_2) = 2$

Target: $\Gamma_1(a_1)$

Desiderata

Theorem (Smith)

An imprimitive metrically homogeneous graph is bipartite or antipodal.

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Problem (Unfinished business)

Give a conditional reduction of the antipodal case.